Lec - 01

Def^n

An Equation which formed by Direct differentiation to its Sol^n is stb "exact diff^n eq^n"

Ex:-

$$x^2dx - y^2dy = 0 \quad \text{(1)} \rightarrow \text{exact}$$

Since its Sol^n is

If we integrate, $\frac{x^3}{3} - \frac{y^3}{3} = c$

$$x^3 - y^3 = 3c \qquad \text{or } 3c = C$$

$$x^3 - y^3 = C \quad \text{— (2)}$$

If we differentiate (2):-

$$3x^2dx - 3y^2dy = 0$$

$$x^2dx - y^2dy = 0$$

(1) Homogeneous

An Eq^n in which is Degree one is term of diff^n eq^n. is called Homogeneous

(2) Linear

An eq^n in which dependent variable and its derivative is of degree one and not multiplied together.

$$\frac{dy}{dx} + Py = Q$$

I.F:- $e^{\int P dx}$

Sol^n :- $y \cdot \text{I.F.} = \int Q \cdot \text{I.F} + c$

(3) ODE

ODE:- An eq^n ~~in~~ which involve single Ind. var. Ex:- $\frac{d^2y}{dx^2} + 2\frac{dy}{dx}$

(4) P.D.E :- which are involved more than Independ...

CT

Que (1)

Ste

Ste

Now

Sol^n -

Exact Differential Equation :–

There is a necessary and sufficient condition for Exact diffⁿ Eqⁿ.

$$\boxed{\text{The diff}^n\ \text{Eq}^n\quad M\,dx + N\,dy = 0 \iff \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

The Solⁿ of Exact diffⁿ is

$$\boxed{\int_{(y\ \text{constant})} M\,dx + \int_{(\text{terms of } N \text{ not containing } x)} N\,dy = C}$$

C.T.

① Solve, $(x^2 - ay)\,dx = (ax - y^2)\,dy$

Step (i) → $(x^2 - ay)\,dx - (ax - y^2)\,dy = 0$

Here,

$$M = x^2 - ay, \qquad N = -(ax - y^2)$$

Step (ii) → $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then it is Exact diffⁿ eqⁿ

$$\frac{\partial M}{\partial y} = -a, \qquad \frac{\partial N}{\partial x} = -a$$

i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then we can say that given diffⁿ eqⁿ is exact.

Now, we will find Solⁿ of diffⁿ eqⁿ

$$= \int_{y\ \text{const}} M\,dx + \int_{x\ \text{not containing}} N\,dy = C$$

Sol^n $\Rightarrow$ $\displaystyle\int_{y\ \text{const.}} (x^2 - ay)\, dx + \int y^2\, dy = c$

$\Rightarrow \dfrac{x^3}{3} - axy + \dfrac{y^3}{3} = c$

$\Rightarrow x^3 - 3axy + y^3 = 3c$

Or $\quad x^3 + y^3 - 3axy = C$, Where $3c = C$ New const

Que (2) $\quad (x^2+y^2-a^2)\,x\,dx + (x^2-y^2-b^2)\,y\,dy = 0$

$M = x^3 + xy^2 - a^2x \quad , N = x^2y - y^3 - b^2y$

Now, $\quad \dfrac{\partial M}{\partial y} = 2xy \quad , \quad \dfrac{\partial N}{\partial x} = 2xy$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, i.e. Given eq. Exact diff eq^n.

Then Sol^n $\Rightarrow$

$\displaystyle\int_{y\ \text{const}} (x^3 + xy^2 - a^2x)\, dx + \int_{x\ \text{not containing}} (-y^3 - b^2y)\, dy = c$

$\dfrac{x^4}{4} + \dfrac{x^2y^2}{2} - \dfrac{a^2x^2}{2} - \dfrac{y^4}{4} - \dfrac{b^2y^2}{2} = C$

$\dfrac{x^4}{4} - \dfrac{y^4}{4} + \dfrac{x^2y^2}{2} - \dfrac{a^2x^2}{2} - \dfrac{b^2y^2}{2} = C$

$x^4 - y^4 + 2x^2y^2 - 2a^2x^2 - 2b^2y^2 = 2C$

Or $\quad x^4 - y^4 + 2x^2y^2 - 2a^2x^2 - 2b^2y^2 = C$

3) $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$

$M = x^2 - 4xy - 2y^2$, $N = y^2 - 4xy - 2x^2$

$\frac{\partial M}{\partial y} = -4x - 4y$, $\frac{\partial N}{\partial x} = -4y - 4x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, i.e. Exact diff^n eq^n.

Sol^n $\longrightarrow$

$$\int_{y\,\text{const.}} (x^2 - 4xy - 2y^2)\,dx + \int_{x\text{ not containing}} y^2\,dy = C$$

$$\Rightarrow \frac{x^3}{3} - 4y\frac{x^2}{2} - 2y^2x + \frac{y^3}{3} = C$$

$$\Rightarrow x^3 - 6x^2y - 6xy^2 + y^3 = 3C$$

or

$$x^3 + y^3 - 6x^2y - 6xy^2 = C$$

$(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0$

$M = x^4 - 2xy^2 + y^4$, $N = -2x^2y + 4xy^3 - \sin y$

$\frac{\partial M}{\partial y} = -4xy + 4y^3$, $\frac{\partial N}{\partial x} = -4xy + 4y^3$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, i.e. Exact diff^n eq^n.

Sol^n $\longrightarrow$

$$\int_{y\,\text{const}} (x^4 - 2xy^2 + y^4)\,dx + \int_{x\text{ not}} -\sin y\,dy = C$$

$$\frac{x^5}{5} - 2y^2\frac{x^2}{2} + y^4x + \cos y = C$$

$$x^5 - 5x^2y^2 + 5xy^4 + 5\cos y = 5C$$

or

$$x^5 - 5x^2y^2 + 5xy^4 + 5\cos y = C$$

(5) $ye^{xy}dx + (xe^{xy} + 2y)\,dy = 0$

$$M = ye^{xy} \quad , \quad N = xe^{xy} + 2y$$

$$\frac{\partial M}{\partial y} = xye^{xy} + e^{xy} \quad , \quad \frac{\partial N}{\partial x} = xye^{xy} + e^{xy}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad , \quad \text{i.e. exact diff eq}^{n}.$$

Sol^{n} $\rightarrow$

$$\int_{y\ \text{const}} ye^{xy}\,dx + \int_{x\ \text{not}} 2y\,dy = c$$

$$\Rightarrow \cancel{y}\frac{e^{xy}}{\cancel{y}} + \cancel{2}\frac{y^2}{\cancel{2}} = c$$

$$\Rightarrow e^{xy} + y^2 = c$$

(6) $(5x^4 + 3x^2y^2 - 2xy^3)\,dx + (2x^3y - 3x^2y^2 - 5y^4)\,dy$

$$M = 5x^4 + 3x^2y^2 - 2xy^3$$

$$N = 2x^3y - 3x^2y^2 - 5y^4$$

$$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2$$

$$\frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad , \quad \text{i.e. Exact diff}^{n}$$

Sol^{n} $\rightarrow$

$$\int_{y\ \text{const.}} (5x^4 + 3x^2y^2 - 2xy^3)\,dx + \int_{x\ \text{not containing}} -5y^4\,dy = c$$

$$\Rightarrow \cancel{5}\frac{x^5}{\cancel{5}} + \frac{\cancel{3}x^3}{\cancel{3}}y^2 - \cancel{2}\frac{x^2}{\cancel{2}}y^3 - \cancel{5}\frac{y^5}{\cancel{5}} = c$$

$$\Rightarrow x^5 + x^3y^2 - x^2y^3 - y^5 = c$$

(7) $(3x^2+6xy^2)\,dx+(6x^2y+4y^3)\,dy=0$

$M = 3x^2+6xy^2$, $N = 6x^2y+4y^3$

$\frac{\partial M}{\partial y} = 12xy$, $\frac{\partial N}{\partial x} = 12xy$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, i.e. exact diff eqn.

Soln $\longrightarrow$

$$\int_{y\ \text{const}} (3x^2+6xy^2)\,dx + \int_{x\ \text{not}} 4y^3\,dy = C$$

$$\Rightarrow \cancel{3}\frac{x^3}{\cancel{3}} + \overset{3}{\cancel{6}}y^2\cdot\frac{x^2}{\cancel{2}} + \cancel{4}\frac{y^4}{\cancel{4}} = C$$

$$\Rightarrow x^3+3y^2x^2+y^4 = C$$

(8) $\frac{2x}{y^3}\,dx + \frac{y^2-3x^2}{y^4}\,dy = 0$

$M = \frac{2x}{y^3}$, $N = \frac{y^2-3x^2}{y^4}$

$\frac{\partial M}{\partial y} = 2x\left(\frac{-3}{y^4}\right)$, $\frac{\partial N}{\partial x} = \frac{-6x}{y^4}$

$= -\frac{6x}{y^4}$,

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, i.e. exact diffn eq

Soln $\longrightarrow$

$$\int_{y\ \text{const}} \left(\frac{2x}{y^3}\right)dx + \int_{x\ \text{not}} \frac{y^2-\cancel{3x^2}}{y^4}\,dy = C$$

$$\Rightarrow \frac{1}{y^3}\left(\frac{\cancel{2}x^2}{\cancel{2}}\right) \cancel{+} - \frac{1}{y} = C$$

$$\Rightarrow \frac{x^2}{y^3} - \frac{1}{y} = C$$

$$\Rightarrow x^2-y^2 = cy^3$$

(9) $y \sin 2x \, dx - (1 + y^2 + \cos^2 x)\, dy = 0$

$M = y \sin 2x \quad , \quad N = -1 - y^2 - \cos^2 x$

$\frac{\partial M}{\partial y} = \sin 2x \quad , \quad \frac{\partial N}{\partial x} = +2\cos x \sin x$

Solⁿ → $\int_{y\ \text{const}} M\,dx + \int_{x\ \text{not}} N\,dy = c$

$\int y \sin 2x \, dx + \int (-1 - y^2)\, dy = c$

$\Rightarrow -y\frac{\cos 2x}{2} - y - \frac{y^3}{3} = c$

$\Rightarrow \frac{-y\cos 2x}{2} - y - \frac{y^3}{3} = c$ Ans.

Or $3y\cos 2x + 6y + 2y^3 = 6c$ or c (New const.)

(10) $(\sec x \tan x \tan y - e^x)\, dx + \sec x \sec^2 y \, dy = 0$

$M = (\sec x \tan x \tan y - e^x)$

$N = (\sec x \sec^2 y)$

$\frac{\partial M}{\partial y} = \sec x \cdot \tan x \cdot \sec^2 y$

$\frac{\partial N}{\partial x} = \sec x \cdot \tan x \cdot \sec^2 y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; i.e. Exact diff eqⁿ

$\int_{y\ \text{const.}} M\,dx + \int_{x\ \text{not containing}} N\,dy = c$

$\int_{y\ \text{const.}} [\sec x \cdot \tan x \cdot \tan y - e^x]\, dx + \int 0 \, dy = c$

$$\sec x \cdot \tan y - e^{x} = c$$

Ans

(11) $(2xy + y - \tan y)\,dx + (x^2 - x\tan^2 y + \sec^2 y)\,dy = 0$

$$M = 2xy + y - \tan y$$

$$N = x^2 - x\tan^2 y + \sec^2 y$$

$$\frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y = 2x - \tan^2 y$$

$(\sec^2 y - 1 = \tan^2 y)$

$$\frac{\partial N}{\partial x} = 2x - \tan^2 y = 2x - \tan^2 y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \text{ i.e. exact diff}^n \text{ eq}^n.$$

Sol^n $\rightarrow$ $\displaystyle\int_{y\ \text{const}} (2xy + y - \tan y)\,dx + \int_{x\ \text{not}} \sec^2 y \,\cancel{\tan^2 y}\,dy = c$

$$\Rightarrow \frac{2x^2}{2} y + xy - x\tan y + \tan y = c$$

$$\Rightarrow x^2 y + xy + (-x + 1)\tan y = c$$

Ans

(12) $[y(1 + 1/x) + \cos y]\,dx + [x + \log x - x\sin y]\,dy = 0$

$$M = y(1 + 1/x) + \cos y$$

$$N = x + \log x - x\sin y$$

$$\frac{\partial M}{\partial y} = (1 + 1/x) - \sin y$$

$$\frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \text{ i.e. exact diff}^n \text{ eq}^n.$$

Sol^n is $\longrightarrow$

$$\int_{y\text{ const}} M\,dx + \int_{x\text{ not containing}} N\,dy = C$$

$$\Rightarrow \int_{y\text{ const}} \left[y(1+1/x) + \cos y\right]dx + \int 0 \; dy =$$

$$\Rightarrow y\left[x + \log x\right] + \cancel{\sin y}\, x\cos y = C$$

Integrating factor →

Some times a differential Eqⁿ which is not exact, can be made so on multiplication by a suitable factor is called "an integrating factor."

Inspection Method

In this connection the following integrable combinations prove quite useful:

$$x\,dy + y\,dx = d(xy)$$

$$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$$

$$\frac{x\,dy - y\,dx}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$$

$$\frac{x\,dy - y\,dx}{y^2} = -d\left(\frac{x}{y}\right)$$

$$\frac{x\,dy - y\,dx}{x^2+y^2} = d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$\frac{x\,dy - y\,dx}{x^2-y^2} = d\left[\frac{1}{2}\log\frac{x+y}{x-y}\right]$$

① ~~$(x^2+y^2)\,dx - 2xy$~~

①

Que ① $x\,dy - y\,dx + a(x^2+y^2)dx = 0$ --- ①

$$[a(x^2+y^2) - y]dx + x\,dy = 0$$

i.e. $M\,dx + N\,dy = 0$

Now, $\frac{\partial M}{\partial y} = 2ay - 1$

$$\frac{\partial N}{\partial x} = 1$$

Here, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ i.e. not exa

Now, we will introduce a suitable factor which can make the exact di eq

eq^n to given eq^n.

We will multiplying by (x^2+y^2) in

$$\frac{x\,dy - y\,dx}{x^2+y^2} + \frac{a(x^2+y^2)\,dx}{x^2+y^2} = 0$$

But $\frac{x\,dy - y\,dx}{x^2+y^2} = d\{\tan^{-1}(y/x)\}$

$$\Rightarrow d\left[\tan^{-1}(y/x)\right] + a\,dx = 0$$

Now By integration;

$$\tan^{-1} y/x + ax = C$$

(2) $y\,dx - x\,dy + \log x\,dx = 0$ --- ①

$$(y + \log x)dx - x\,dy = 0$$

$\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = -1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, i.e not exact we multiply by x^2 : — (in ①)

$$\frac{y\,dx - x\,dy}{x^2} + \frac{\log x}{x^2}\,dx = 0$$

$$-d(y/x) + \frac{\log x}{x^2}\,dx = 0$$

by integration;

$$-\int d(y/x) + \int \frac{\log x}{x^2}\,dx = C$$

$$-y/x + \int \underset{\mathrm{I}}{\log x} \cdot \underset{\mathrm{II}}{x^{-2}}\,dx = C$$

$$\Rightarrow -y/x + \log x\left(\frac{-1}{x}\right) - \int \frac{1}{x}\left(\frac{-1}{x}\right)dx = C$$

$$\Rightarrow \frac{-y}{x} - \frac{\log x}{x} + \int \frac{1}{x^2}\,dx = C$$

$$\Rightarrow \frac{-y}{x} - \frac{\log x}{x} - \frac{1}{x} = C$$

$$\Rightarrow -y - \log x - 1 = Cx$$

$x\,dy - y\,dx + 2x^3\,dx = 0$ ----①

$$(-y + 2x^3)\,dx + x\,dy = 0$$

$$\frac{\partial M}{\partial y} = -1 \quad , \quad \frac{\partial N}{\partial x} = 1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$; not exact.

I.F. $\longrightarrow$ $1/x^2$.

$$\frac{x\,dy - y\,dx}{x^2} + 2x\,dx = 0$$

By integration;

$$d(y/x) + 2\frac{x^2}{2} = C$$

$$y/x + x^2 = C$$

(4) $$x\,dx + y\,dy = \frac{a^2(x\,dy - y\,dx)}{x^2+y^2}$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$ i.e. not exact.

$$x\,dx + y\,dy = a^2\, d(\tan^{-1} y/x)$$

By integration,

$$\frac{x^2}{2} + \frac{y^2}{2} = a^2 \tan^{-1} y/x + c$$

Ans

I.F

Homogeneous Equations

If $M\,dx + N\,dy = 0$ be a Homogeneous eqn in x and y then $\frac{1}{Mx+Ny}$ is an integrating factor

$$Mx + Ny \neq 0$$

(1) $$\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$$

$$xy^2\,dy = (x^3+y^3)\,dx$$

$$(x^3+y^3)\,dx - xy^2\,dy = 0 \quad \text{--- (1)}$$

$$\frac{\partial M}{\partial y} = 3y^2 \quad , \quad \frac{\partial N}{\partial x} = -y^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

By other book

since

Not exact.

To make it exact, we will multiply an I.F.

$$\frac{1}{Mx+Ny}$$ since given diff^n eq^n is H.E.

monomial term

I.F

$$\Rightarrow \frac{1}{(x^3+y^3)x+(-xy^2)y}$$

$$\Rightarrow \frac{1}{x^4+\cancel{xy^3}-\cancel{xy^3}} = \frac{1}{x^4}$$

Now, By ① : —

$$\left(\frac{x^3+y^3}{x^4}\right)dx - \frac{xy^2}{x^4}dy = 0$$

On integration,

$$\int_{y\ const.}\left(\frac{1}{x}+\frac{y^3}{x^4}\right)dx - \int_{x\ not}\frac{\cancel{y^2}\ 0}{\cancel{x^3}}dy = c$$

$$\log x + y^3\left(-\frac{1}{3x^3}\right) - \cancel{\frac{1}{x^3}}\ \cancel{\frac{y^3}{3}} = c$$

$$\log x - \frac{y^3}{3x^3} - 0\ \cancel{\frac{y^3}{3x^3}} = c$$

$$\log x - \frac{1}{3}\frac{y^3}{x^3} = c.$$ Ans

By other Book

(2) $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

$$\frac{\partial M}{\partial y} = x^2 - 4xy$$

$$\frac{\partial N}{\partial x} = -3x^2 + 6xy$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$, not exact

since, given diff^n eq^n is Homogeneous diff^n eq^n.

Then I.F. $\frac{1}{Mx+Ny}$

$\Rightarrow \frac{1}{(x^2y-2xy^2)x-(x^3-3x^2y)y}$

$\Rightarrow \frac{1}{\cancel{x^3y}-2x^2y^2-\cancel{x^3y}+3x^2y^2}$

$\Rightarrow \frac{1}{x^2y^2} = \text{I.F.}$

$$\left(\frac{x^2y-2xy^2}{x^2y^2}\right)dx - \left(\frac{x^3-3x^2y}{x^2y^2}\right)dy =$$

$$\int_{y\text{ const}}\left(\frac{1}{y}-\frac{2}{x}\right)dx + \int_{x\text{ not}}\frac{3}{y}\,dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$\frac{x}{y} - \log x^2 + \log y^3 = c$$

$$\frac{x}{y} + \log\frac{y^3}{x^2} = c$$

(3) $x^2y\,dx - (x^3+y^3)\,dy = 0 \quad \cdots (1)$

$$\frac{\partial M}{\partial y} = x^2$$

$$\frac{\partial N}{\partial x} = -3x^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

not exact

since given diffⁿ eqⁿ is Homo. eqⁿ then

I.F. is $\frac{1}{Mx+Ny}$

$$\Rightarrow \frac{1}{(x^2y)x + (-x^3-y^3)y}$$

$$\Rightarrow \frac{1}{x^3y - x^3y - y^4} = \frac{-1}{y^4}$$

By (1):—

$$-\frac{x^2y}{y^4}dx - \left\{-\frac{(x^3+y^3)}{y^4}\right\}dy = 0$$

$$-\frac{x^2}{y^3}dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0$$

By integration;

$$-\int_{y\ const} \frac{x^2}{y^3}dx + \int_{x\ not} \frac{1}{y}dy = C$$

$$-\frac{x^3}{3y^3} + \log y = C \quad \text{Ans}$$

Que :- $(3xy^2 - y^3)dx - (2x^2y - xy^2)dy = 0$

Ans :- $3\log x + \frac{y}{x} - 2\log y = C$

at least one term is $f(x,y)$ after making this type of form.

I.F. for an equation of the type

$$\boxed{f_1(xy)\, y\, dx + f_2(xy)\, x\, dy = 0}$$

If the eq^n $M dx + N dy = 0$ be of this form, then $\dfrac{1}{Mx - Ny}$ is an integrating factor. $Mx - Ny \neq 0$

Que →

$$(x^3y^2 + x)\, dy + (x^2y^3 - y)\, dx = 0.$$

$$(x^2y^2 + 1)x\, dy + (x^2y^2 - 1)\, y\, dx = 0$$

First we will check the diff^n is exact or not.

we get $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

i.e. not exact.

Then we can say that The diff^n eq^n is of the form

$$f_1(x,y)\, y\, dx + f_2(xy)\, x\, dy = 0$$

Then I.F. is $\dfrac{1}{Mx - Ny}$

By ① →

$$\frac{1}{(x^2y^3 - y)x - (x^3y^2 + x)y}$$

$$= \frac{1}{x^3y^3 - xy - x^3y^3 - xy}$$

$$\text{I.F.} = \frac{1}{-2xy}$$

(c) Since, we can write all constant terms as $x^0 y^0$

By Multiplying :

$$\left(\frac{x^2y^3 - y}{-2xy}\right)dx + \left(\frac{x^3y^2 + x}{-2xy}\right)dy = 0$$

$$\cancel{-\frac{xy^2}{2} + \frac{1}{2x}}$$

Then the Solⁿ is

$$\int\left(\frac{x^2y^3 - y}{-2xy}\right)dx + \int -\frac{1}{2y}\,dy = c$$

$$\int\left(\frac{-xy^2}{2} + \frac{1}{2x}\right)dx - \frac{1}{2}\int \frac{1}{y}\,dy = c$$

$$-\frac{x^2y^2}{4} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$-\frac{x^2y^2}{4} + \log\frac{\sqrt{x}}{\sqrt{y}} = c$$

Extra Ques

Ques $(1+xy)y\,dx + (1-xy)x\,dy = 0$

Ques → $(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}, \text{ not exact.}$$

Then I.F $\dfrac{1}{Mx - Ny}$

$$\frac{1}{(x^2y^3 + xy^2 + y)x - (x^3y^2 - x^2y + x)y}$$

$$\frac{1}{x^3y^3 + x^2y^2 + xy - x^3y^3 + x^2y^2 - xy}$$

$$\frac{1}{2x^2y^2}$$

By Multiplying $\dfrac{1}{2x^2y^2}$

$$\Rightarrow \frac{1}{2}\left[xy + \log x - \frac{y}{x}\right] - \frac{1}{2}\log y = c$$

$$\left(\frac{x^2y^3+xy^2+y}{2x^2y^2}\right)_M dx + \left(\frac{x^3y^2-x^2y+x}{2x^2y^2}\right)_N$$

Now Sol^n is

$$\frac{1}{2}\int\left(\frac{y}{2}+\frac{1}{x}+\frac{1}{x^2y}\right)dx+\int\left(\frac{-1}{2y}\right)dy=$$

$$\frac{1}{2}\left[xy+\log x-\frac{y}{x}\right]-\frac{1}{2}\log y=c$$

Ans

(4) In the equation $Mdx + Ndy = 0$

(a) if $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} =$ only $f(x)$

then I.F. is $e^{\int f(x)\,dx}$.

(b) if $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} =$ only $f(y)$

Then I.F. is $e^{\int f(y)\,dy}$.

—o—

Que → $(y^4+2y)\,dx + (xy^3+2y^4-4x)\,dy = 0$

Firstly, $\dfrac{\partial M}{\partial y} = 4y^3+2$

$\dfrac{\partial N}{\partial x} = y^3-4$

Now, $-\dfrac{\partial M}{\partial y} + \dfrac{\partial N}{\partial x} = -(4y^3+2-y^3+4$

$= -(3y^3+6)$

$= -3(y^3+2)$

I.F. $\dfrac{-\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}}{M} = \dfrac{-3(y^3+2)}{y(y^3+2)}$

$= \dfrac{-3}{y}$ ~~= I.F.~~

Then I.F. $= e^{\int -3/y\,dy} = e^{-3\log y} = e^{\log y^{-3}} = y^{-3}$

~~Then the Soln is~~

$\left(\dfrac{y^4+2y}{y^3}\right)dx + \left(\dfrac{xy^3+2y^4-4x}{y^3}\right)dy = 0$

Now we will check that the eq^n is exact or not.

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[y + \frac{2}{y^2}\right]$$

$$= \left(1 + \frac{-4}{y^3}\right)$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[x + 2y - \frac{4x}{y^3}\right]$$

$$= \left(1 - \frac{4}{y^3}\right)$$

i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ = Exact diff^{n}e

Sol^{n} $\longrightarrow$ $\int_{y\,const}\left(y + \frac{2}{y^2}\right)dx + \int 2y\,dy = c$

$$yx + \frac{2}{y^2}x + \frac{2y^2}{2} = c$$

$$yx + \frac{2x}{y^2} + y^2 = c$$

The

Que $\rightarrow$ $(3xy - 2ay^2)dx + (x^2 - 2axy)dy = 0$

$$\frac{\partial M}{\partial y} = 3x - 4ay$$

$$\frac{\partial N}{\partial x} = 2x - 2ay$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ i.e. not ex

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = x - 4ay + 2ay$$

$$= x - 2ay$$

Now,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{x - 2ay}{x^2 - 2axy}$$

$$= \frac{x - 2ay}{x(x-2ay)} = 1/x = f(x)$$

Then I.F. $e^{\int 1/x \, dx} = e^{\log x} = x$

~~Then~~ $\left(\dfrac{3xy - 2ay^2}{x}\right) dx + \left(\dfrac{x^2 - 2axy}{x}\right) dy = 0$

$$\Rightarrow \underbrace{\left(3y - 2a\frac{y^2}{x}\right)}_{M} dx + \underbrace{(x - 2ay)}_{N} dy = 0$$

$$\frac{\partial M}{\partial y} = 3 - \frac{4ay}{x}$$

& $\dfrac{\partial N}{\partial x} = 1$

Then, $(3xy - 2ay^2)x \, dx + (x^2 - 2axy)x \, dy = 0$

$$\underbrace{(3x^2y - 2axy^2)}_{M} dx + \underbrace{(x^3 - 2ax^2y)}_{N} dy = 0$$

$$\frac{\partial M}{\partial y} = 3x^2 - 4axy, \quad \frac{\partial N}{\partial x} = 3x^2 - 4axy$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$, i.e. exact diff^n eq^n.

The Sol^n is

$$\int_{y\ \text{const}} (3x^2y - 2axy^2) \, dy + \int 0 \, dy = c$$

$$3\frac{x^3}{3}y - 2a\frac{x^2}{2}y^2 = c$$

$$x^3y - ax^2y^2 = c$$

Ans

$$\cancel{\frac{1}{3x^2y - 2axy^2 + x^2y - 2axy^2}} = \cancel{\frac{1}{4x^2y - 4axy^2}}$$

$$x^4 \frac{dy}{dx} + x^3 y + \operatorname{cosec} xy = 0$$

$$x^4 \frac{dy}{dx} + x^3 y + \frac{1}{\sin(xy)} = 0$$

$$x^4 \sin(xy) \frac{dy}{dx} + \sin(xy)\, x^3 y + 1 = 0$$

$$x^4 \sin(xy)\, dy + \sin(xy)\, x^3 y\, dx + dx = 0$$

$$x^4 \sin(xy)\, dy + [\sin(xy)\, x^3 y + 1]\, dx$$

$$M = \sin(xy)\, x^3 y + 1$$

$$N = x^4 \sin(xy)$$

$$\frac{\partial M}{\partial y} = x^3 [yx\cos(xy) + \sin(xy)]$$

$$\frac{\partial N}{\partial x} = x^4 y\cos(xy) + 4x^3 \sin(xy)$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$, not exact d.E.

Then $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = yx^4 \cos xy + x^3 \sin(xy) - x^4 y\cos(xy) - 4x^3 \sin(xy)$

$= -3x^3 \sin(xy)$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-3x^3 \sin(xy)}{x^4 \sin(xy)}$$

$$= -\frac{3}{x} = f(x)$$

Then I.F. $e^{\int -3/x \, dx} = e^{-3\log x}$

$\Rightarrow e^{\log x^{-3}} = x^{-3}$

Multiplying By x^{-3}:

we get, $\dfrac{\sin(xy)\, x^3 y + 1}{x^3} dx + \dfrac{x^4 \sin(xy)}{x^3} = 0$

$\Rightarrow \underbrace{\left(y\sin(xy) + \frac{1}{x^3}\right)}_{M} dx + \underbrace{x \sin xy}_{N} = 0$

$\dfrac{\partial M}{\partial y} = \sin xy \; , \; \dfrac{\partial N}{\partial x} = \sin xy$

$\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, i.e. exact.

Then the Solⁿ is

$\int_{y\ \text{const.}} \left[y\sin(xy) + \frac{1}{x^3}\right] dx + \int 0\, dy = c$

$-\dfrac{\cancel{y}\cos(xy)}{\cancel{y}} - \dfrac{1}{2x^2} = c$

$\cos(xy) + \dfrac{1}{2x^2} = c'$

Que → $(x^4 e^x - 2mxy^2)\,dx + 2mx^2 y\, dy = 0$

By check the eqⁿ, $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ i.e. not exact.

then, $\dfrac{\partial M}{\partial y} = -4mxy \; , \; \dfrac{\partial N}{\partial x} = 4mxy$

$\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = -4mxy - 4mxy$

$= -8mxy$

$\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \dfrac{-8mxy}{2mx^2 y} = \dfrac{-4}{x} = f(x)$

I.F. $\rightarrow e^{\int -\frac{4}{x}dx} = e^{-4\log x} = e^{\log \dots} = x^{\dots}$

By Multiplying through out the ~~of~~ eq^n by $1/x^4$

$$\left(\frac{x^4e^x - 2mxy^2}{x^4}\right)dx + \frac{2mx^2y}{x^4}\,dy = 0$$

$$\underbrace{\left(e^x - \frac{2my^2}{x^3}\right)}_{M}dx + \underbrace{\frac{2my}{x^2}}_{N}dy = 0$$

$$\frac{\partial M}{\partial y} = -\frac{4my}{x^3}, \quad \frac{\partial N}{\partial x} = 2my\,x^{-2\dots} = \frac{-4my}{x^3}$$

i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ i.e. exact e...

Then the sol^n is

$$\int_{y\ \text{const}}\left(e^x - \frac{2my^2}{x^3}\right)dx + \int 0\,dy = C$$

$$\Rightarrow \left[e^x - \cancel{2}my^2 \times \frac{-1}{\cancel{2}x^2}\right] = C$$

$$\Rightarrow e^x + \frac{my^2}{x^2} = C$$

Ans

1) Que :- $(xy^2 - e^{1/x^3})\,dx - x^2y\,dy = 0$ Imp

2) HW Que :- $2xy\,dy - (x^2+y^2+1)\,dx = 0$

3) HW Que $(x^2+y^2)dx - 2xy\,dy = 0$

HW Ans $x - \frac{y^2}{x} = C$

4) HW Que $(xy^3+y)dx + 2(x^2y^2+x+y^4)dy = 0$

Ans $\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{6} = C$

Que → $y\,dx - x\,dy + 3x^2y^2e^{x^3}dx = 0$

(By inspection Method)

first we will check that the given diffⁿ eqⁿ is exact or not.

we get $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ i.e. not exact.

Now, $y\,dx - x\,dy + 3x^2y^2e^{x^3}dx = 0$

Multiplying by y^2:–

$$\frac{y\,dx - x\,dy}{y^2} + 3x^2e^{x^3}dx = 0$$

$$d\left[\frac{x}{y}\right] + 3x^2e^{x^3}dx = 0$$

On integrating;

$$\int d(x/y) + 3\int x^2e^{x^3}dx = 0$$

$$\frac{x}{y} + e^{x^3} = C$$

since, $\int x^2 . e^{x^3}dx$

let $x^3 = t$, $3x^2dx = dt$, $x^2dx = \frac{1}{3}dt$

$\int e^t dt$

$\Rightarrow e^t$

$\Rightarrow e^{x^3}$

Que: → $(xy^2 - e^{1/x^3})dx - x^2y\,dy = 0$

Solve: → $\frac{\partial M}{\partial y} = 2xy$, $\frac{\partial N}{\partial x} = -2xy$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2xy + 2xy}{-x^2y} = \frac{-4}{x}$$

I.F. :– $e^{\int -4/x} = x^{-4}$

Then $\left(\frac{xy^2 - e^{1/x^3}}{x^4}\right)dx + \left(\frac{-x^2y}{x^4}\right)dy = 0$

$$\int\left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right)dx = C$$

$$\Rightarrow y^2 . \frac{x^{-2}}{-2} - \int \frac{e^{1/x^3}}{x^4}dx = C$$

let $x^{-3} = t \Rightarrow -3x^{-4}dx = dt$

$\Rightarrow \frac{-y^2}{2x^2} + \int e^t \frac{dt}{3} = C$

$\Rightarrow \frac{-y^2}{2x^2} + \frac{e^t}{3} = C$

$\Rightarrow \frac{-y^2}{2x^2} + \frac{e^{x^{-3}}}{3} = C$

Que → $(y^2+2x^2y)\,dx+(2x^3-xy)\,dy=0$

V^{th} - Type

$x^a y^b(my\,dx+nx\,dy)+x^{a'}y^{b'}(m'y\,dx+n'x\,dy)$

I.F. → $\underline{\underline{x^h y^k}}$

Where, $\dfrac{a+h+1}{m}=\dfrac{b+k+1}{n}$

$\dfrac{a'+h+1}{m'}=\dfrac{b'+k+1}{n'}$

At last ②

Que → $(y^2+2x^2y)\,dx+(2x^3-xy)\,dy=0$

$\cancel{y(y+2x^2)\,dx+x(2x^2-y)\,dy=0}$

$\cancel{x^0y^1(y+2x^2)\,dx+x^1y^0(2x^2-y)\,dy=0}$

$\cancel{a=0,\ b=1 \quad a'=}$

$y^2dx+2x^2y\,dx+2x^3dy-xy\,dy=0$

$[y\,dx-x\,dy]\,y\,dx+2x^2[y\,dx+x\,dy]=0$

$x^0y^1[y\,dx-x\,dy]\,dx+x^2y^0[2y\,dx+2x\,dy]=$

By comparing V^{th} type ;

$a=0$	$a'=2$
$b=1$	$b'=0$
$m=1$	$m'=2$
$n=-1$	$n'=2$

Now we will find

I.F. $\rightarrow x^h y^k$

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}$$

$$\frac{0+h+1}{1} = \frac{1+k+1}{-1}$$

$$-(h+1) = 2+k$$

$$-(h+1) = 2+k$$

$$k+h = -3 \quad \text{---} (1)$$

and $$\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

$$\frac{2+h+1}{2} = \frac{0+k+1}{2}$$

$$h+3 = k+1$$

$$h-k = -2 \quad \text{--} (2)$$

By (1) & (2) :- $$h+k = -3$$
$$h-k = -2$$
$$\Rightarrow \left.\begin{aligned} h &= -5/2 \\ k &= -1/2 \end{aligned}\right\}$$

Then, I.F. $\rightarrow x^h y^k$

$$x^{-5/2} \cdot y^{-1/2}$$

By multiplying

$$\frac{(y^2 + 2x^2y)dx}{x^{5/2} \cdot y^{1/2}} + \frac{2x^3 - xy}{x^{5/2} y^{1/2}} dy = 0$$

Sol^n $\rightarrow$ $$\int (y^{3/2} \cdot x^{-5/2} + 2x^{-1/2} y^{1/2}) dx + 0 = c$$

$$\frac{-2}{3} \frac{y^{3/2}}{x^{3/2}} + 4y^{1/2} x^{1/2} = c$$

(2)

Que → $(2y\,dx + 3x\,dy) + 2xy(3y\,dx + 4x\,dy) = 0$

$x^0y^0(2y\,dx + 3x\,dy) + x^1y^1(6y\,dx + 8x\,dy) = 0$

By Comparing with

$x^a y^b[my\,dx + nx\,dy] + x^{a'}y^{b'}[m'y\,dx + n'x\,dy] = 0$

$a = 0,\quad b = 0,\quad m = 2,\quad n = 3$

$a' = 1,\quad b' = 1,\quad m' = 6,\quad n' = 8$

Now

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}$$

$$\frac{h+1}{2} = \frac{k+1}{3}$$

$$3h+3 = 2k+2$$

$$3h-2k = -1 \quad \text{---} (1)$$

$$\frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

$$\frac{1+h+1}{6} = \frac{1+k+1}{8}$$

$$8[2+h] = 6[k+2]$$

$$8h+16 = 6k+12$$

$$8h-6k = -4$$

$$4h-3k = -2 \quad \text{---} (2)$$

By (1) & (2) :-

$$3h-2k = -1 \quad \times 3$$

$$4h-3k = -2 \quad \times 2$$

$$9h - 6k = -3$$

$$8h - 6k = -4$$

$$-\quad +\quad +$$

$$h = 1$$

$\boxed{h=1}$ & $\cancel{3h-2k=1}$

$\cancel{3-2k=1}$

$\cancel{2=2k \Rightarrow k=1}$

By, $3h-2k=-1$

$3-2k=-1$

$-2k=-4$

$\boxed{k=2}$

Then I.F. $\underline{x^1y^2}$

~~2ydx+3xdx~~ First we will arrange the eqⁿ in the form of

$Mdx+Ndy=0$

$(2y+6xy^2)dx + (3x+8x^2y)dy=0$

Now multiplying by xy^2.

$$\cancel{\left(\frac{2y+6xy^2}{xy^2}\right)dx + \left(\frac{3x+8x^2y}{xy^2}\right)dy=0}$$

~~Solⁿ~~ $\Rightarrow$

$$\cancel{\int\left(\frac{2y+6xy^2}{xy^2}\right)dx + \int\frac{3}{y^2}dy = C}$$

$$xy^2\left[(2y+6xy^2)\right]dx + xy^2\left[3x+8x^2y\right]dy=0$$

$$\left[2xy^3+6x^2y^4\right]dx + \left[3x^2y^2+8x^3y^3\right]dy=0$$

Now, Solⁿ $\longrightarrow$

$$\int_{y\text{ const}}\left[2xy^3+6x^2y^4\right]dx + \int 0\,dy = C$$

$$\frac{\cancel{2}x^2}{\cancel{2}}y^3 + \frac{\overset{2}{\cancel{6}}x^3}{\cancel{3}}y^4 = C$$

$$x^2y^3+2x^3y^4=C$$

When we have A diff^n eq^n of high degree, Then we will solve By some other Meth.

Solvable for p

Steps :– (I) First we will arange the
(II) Factorised the equation.
(III) Equating to zero.
(IV) Then again change P into $\frac{dy}{dx}$
(V) Then we will get general Sol^n of the eq^n.

Que– $f(x,y,y')$ or $f(x,y,\frac{dy}{dx})$ or $f(x,y,p)$

and Sol^n – $f(x,y,c)$ (family of curves)

let $y = x + c$, if $c = 0$ then $y = x$
If $c = 1$, then $y = x+1$ i.e curves dep upon c, which exist only in General sol^n But in Sol^n c not exist then we get envolope of curves i.e closed curve which give Solution

Que → $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$

let $\frac{dy}{dx} = p$

$$yp^2 + (x-y)p - x = 0$$

$$yp^2 + xp - yp - x = 0$$

$$yp[p-1] + x[p-1] = 0$$

$$(yp+x)(p-1) = 0$$

$$yp + x = 0$$

$$y\frac{dy}{dx} + x = 0$$

$$y\,dy + x\,dx = 0$$

~~$y\,dy + x\,dx = 0$~~

on integration

~~$yx + xy = c$~~

$$\frac{y^2}{2} + \frac{x^2}{2} = c$$

$$y^2 + x^2 = 2c \Rightarrow (y^2 + x^2 - 2c) = 0$$

$$p - 1 = 0$$

$$\frac{dy}{dx} - 1 = 0$$

$$dy - dx = 0$$

on integ.

$$y - x = c$$

$$(y - x - c) = 0$$

Then the general Sol^n is

$$(y-x-c)(y^2+x^2-2c) = 0$$

Ans

Que → $P(P+y) = x(x+y)$

$$p^2 + py - x^2 - xy = 0$$

$$(p^2 - x^2) + y(p - x) = 0$$

$$(p+x)(p-x) + y(p-x) = 0$$

$$(p-x)[p+x+y] = 0$$

$$p - x = 0$$

$$\frac{dy}{dx} - x = 0$$

$$dy - x\,dx = 0$$

On integrating,

$$y - \frac{x^2}{2} = c$$

$$2y - x^2 = 2c \Rightarrow 2y - x^2 - 2c = 0$$

$$\cancel{p - x + y = 0}$$

$$\cancel{\frac{dy}{dx} - x + y = 0}$$

$$\cancel{dy - x\,dx + y\,dx = 0}$$

~~on integration,~~

$$\cancel{y - \frac{x^2}{2} + yx = c}$$

~~Then The general Soln ;~~

$$\cancel{(2y + x^2 - 2c)\left(y + yx - \frac{x^2}{2} - c\right) = 0}$$

$$p + x + y = 0$$

$p + y = -x$ which is linear eqn.

I.F. ⇒ $e^{\int P\,dx} = e^{\int 1\,dx} = e^x$

Then Soln is

$$y \cdot I.F. = \int Q \cdot I.F. + c$$

$$y \cdot e^x = \int (-x) \cdot e^x\,dx + c$$

$$\cancel{ye^x = (-x)e^x + e^x(-1) + c}$$

$$\cancel{= -xe^x - e^x + c}$$

$$ye^x = -[xe^x - 1 \cdot e^x] + c$$

$$ye^x + xe^x - e^x = c$$

Then the general Sol^n is

$$(2y - x^2 - 2c)(ye^x + xe^x - e^x - c) = 0$$

Ans

Que → * $y = x\left[p + \sqrt{1+p^2}\right]$

*

$$y = xp + x\sqrt{1+p^2}$$

$$y - xp = x\sqrt{1+p^2}$$

Squaring Both Sides ;

$$(y - xp)^2 = x^2(1+p^2)$$

$$y^2 - 2xyp + \cancel{x^2p^2} = x^2 + \cancel{x^2p^2}$$

$$y^2 - 2xyp - x^2 = 0$$

$$y^2 - 2xy\frac{dy}{dx} - x^2 = 0$$

$$y^2 - x^2 = 2xy\frac{dy}{dx}$$

Then $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy}$ —— (1)

This is a Homogeneous eq^n.

Let

$$y = vx$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{—— (2)}$$

By (1) & (2) : ——

$$v + x\frac{dv}{dx} = \frac{y^2 - x^2}{2xy}$$

$$v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x(vx)}$$

Put ($vx = y$)

$$v + x\frac{dv}{dx} = \frac{\cancel{x^2}(v^2 - 1)}{2\cancel{x^2}v}$$

$(y^2 - x^2)\,dx - 2xy\,dy = 0$

$\dfrac{\partial M}{\partial y} = 2y$

$\dfrac{\partial N}{\partial x} = -2y$

$\dfrac{2y + 2y}{-2x}$

$$x\frac{dv}{dx} = \frac{v^2-1}{2v} - v$$

$$x\frac{dv}{dx} = \frac{v^2-1-2v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{-v^2-1}{2v}$$

$$-\frac{2v\,dv}{v^2+1} = \frac{dx}{x}$$

On integration;

$$-\int \frac{2v\,dv}{v^2+1} = \int \frac{dx}{x}$$

$\{$ let $v^2+1=t \Rightarrow 2v\,dv = dt$ $\}$

$$-\int \frac{dt}{t} = \int \frac{dx}{x} + c$$

$$-\log t = \log x + c$$

$$-\log(v^2+1) = \log x + c$$

$$\log x + \log(v^2+1) = c'$$

$$\log x \times (v^2+1) = c'$$

$$\log x \times \left\{\frac{y^2}{x^2}+1\right\} = c'$$

$$\log\left[\frac{x^2+y^2}{x}\right] = c'$$

Taking exponential :—

$$e^{\log\left(\frac{x^2+y^2}{x}\right)} = e^{c'}$$

then $\dfrac{x^2+y^2}{x} = C$

$$x^2+y^2 = Cx$$

OR

$\log x + \log(v^2+1) = \log c$ ← Integration constant

$\log x(v^2+1) = \log c$

$x(v^2+1) = c$

$x\left[\frac{y^2}{x^2}+1\right] = c$

$\frac{x^2+y^2}{x} = c$

$x^2+y^2 = cx$

$$xy\left(\frac{dy}{dx}\right)^2 - (x^2+y^2)\frac{dy}{dx} + xy = 0$$ (5)

Put $\frac{dy}{dx} = p$

$$xyp^2 - (x^2+y^2)p + xy = 0$$

$$xyp^2 - x^2p - y^2p + xy = 0$$

$$yp[xp-y] - x[xp-y] = 0$$

$$(yp-x)(xp-y) = 0$$

$xp - y = 0$

$$x\frac{dy}{dx} - y = 0$$

$$\frac{x\,dy}{dx} = y$$

Then $\frac{dy}{y} = \frac{dx}{x}$

On integration;

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\log y = \log x + c$$

$$\log y/x = c$$

or $y/x - c' = 0$

$yp - x = 0$

$$y\frac{dy}{dx} = x$$

$$y\,dy = x\,dx$$

On integration;

$$\int y\,dy = \int x\,dx + c$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c$$

$$\frac{y^2}{2} - \frac{x^2}{2} = c$$

$$x^2 - y^2 = 2c$$

~~or~~ $x^2 - y^2 = c$

Then the solⁿ is

$$\left(\frac{y}{x} - c'\right)\left(x^2 - y^2 - 2c\right) = 0$$

$$p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$$

$$(p^3 - y^2p^2) + (2xp^2 - 2xy^2p) = 0$$

$$p^2[p - y^2] + 2xp\,[p - y^2] = 0$$

$$(p^2 + 2xp)\,(p - y^2) = 0$$

$$p^2 + 2xp = 0$$

$$\cancel{\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} = 0}$$

$$\cancel{\frac{dy}{dx}} \quad p\,[p + 2x] = 0$$

$p = 0$ and

$$p + 2x = 0$$

$$\frac{dy}{dx} = -2x$$

$$dy = -2x\,dx$$

on integration,

$$\int dy = -2\int x\,dx + c$$

$$y = -2\frac{x^2}{2} + c$$

$$y + x^2 = c$$

$$(y + x^2 - c) = 0 \quad \text{(ii)}$$

$$p - y^2 = 0$$

$$\frac{dy}{dx} - y^2 = 0$$

$$\frac{dy}{dx} = y^2$$

$$\int \frac{dy}{y^2} = \int dx + c$$

$$\frac{y^{-2+1}}{-2+1} = x + c$$

$$\frac{y^{-1}}{-1} - x - c = 0$$

$$\left(-\frac{1}{y} - x - c\right) = 0 \quad \text{(iii)}$$

Then the Solⁿ is

$$(y - c)(y + x^2 - c)(-1/y - x - c) = 0$$

$$(2x^2 + 3y^2 - 7)\,x\,dx - (3x^2 + 2y^2 - 8)\,y\,dy = 0$$

* Solvable for y *

In This Method our main Purpose is to ~~...~~ change $f(x,y)$ in $f(x,P)=y$.

$$y = f(x,p)$$

Step:- ① diffⁿ w.r.t. x :- $\frac{dy}{dx}$

② Taking $\frac{dy}{dx} = p$

③ Solve the eqⁿ and integ it.

④ Then General Solⁿ is

$x = F_1(P,c)$ not involving y , $y = F_2(P,c)$ not involving x

---

Que → Try this at last

$$y = x + a\tan^{-1}p \quad \text{--- ①}$$

diffⁿ w.r.t. x :-

$$\frac{dy}{dx} = 1 + a\frac{1}{1+p^2}\frac{dp}{dx}$$

Put $p = dy/dx$

$$p = 1 + a\frac{1}{1+p^2}\frac{dp}{dx}$$

$$p-1 = \frac{a}{1+p^2}\frac{dp}{dx}$$

or $$\frac{dp}{dx} = \frac{(p-1)(1+p^2)}{a}$$

or $$dx = \frac{a\,dp}{(p-1)(1+p^2)}$$

Solve the eqⁿ By using Partial fraction :

$$\frac{a}{(p-1)(1+p^2)} = \frac{A}{p-1} + \frac{Bp+C}{1+p^2}$$

$$a = A(1+p^2) + (p-1)(Bp+C)$$

$$a = A + Ap^2 + Bp^2 + pC - Bp - C$$

$$a = (A+B)p^2 + p(C-B) + A - C$$

$A + B = 0$, $C - B = 0$ & $A - C = a$ — ①

By

$$A + \cancel{B} = 0$$
$$C - \cancel{B} = 0$$
$$\overline{A + C = 0} \quad \text{— ②}$$

BY ① & ② :-

$$A - \cancel{C} = a$$
$$A + \cancel{C} = 0$$
$$\overline{2A = a \Rightarrow \boxed{A = a/2}}$$

then $\underline{B = -a/2}$ & $\underline{C = -a/2}$

Then we get

$$dx = \left(\frac{a}{2(p-1)} - \frac{a/2\, p - a/2}{1+p^2} \right) dp$$

$$= \frac{a}{2}\left[\frac{1}{p-1} - \frac{p}{1+p^2} - \frac{1}{p^2+1} \right] dp$$

On integration;

$$\int dx = \frac{a}{2}\left[\int \frac{1}{p-1}\, dp - \int \frac{p}{1+p^2}\, dp - \int \frac{1}{p^2+1}\, dp \right]$$

$$\int dx = \frac{a}{2}\left[\log(p-1) - \frac{1}{2}\log(1+p^2) - \tan^{-1}p \right] + C$$

$$x = \frac{a}{2}\left[\log(p-1) - \log\sqrt{1+p^2} - \tan^{-1}p \right] + c$$

$$x = \frac{a}{2}\left[\log\frac{p-1}{\sqrt{1+p^2}} - \tan^{-1}p \right] + c$$

Now putting the value of x in ①

$$y = \frac{a}{2}\left[\log\frac{p-1}{\sqrt{1+p^2}} - \tan^{-1}p \right] + a\tan^{-1}p + c$$

The General solⁿ is →

$$\left. \begin{array}{l} x = \text{- - - -} \\ y = \text{- - - -} \end{array} \right\}$$

Lec-(06)

Solvable for y

$$y + px = x^4p^2 \quad \text{--- (1)}$$

diff^n w.r.t. x:—

$$\cancel{\frac{dy}{dx} + p = 4x^3p^2}$$

$$\frac{dy}{dx} + \left(p \cdot 1 + x\frac{dp}{dx}\right) = \left(2x^4p\frac{dp}{dx} + 4x^3p^2\right)$$

Put $p=$

$$\frac{dy}{dx} = 2x^4p\frac{dp}{dx} + 4x^3p^2 - \left(p + x\frac{dp}{dx}\right)$$

dy/dx

$$p = 2x^4p\frac{dp}{dx} + 4x^3p^2 - \left(p + x\frac{dp}{dx}\right)$$

$$p = (4x^3p^2 - p) + \left(2x^4p\frac{dp}{dx} - x\frac{dp}{dx}\right)$$

$$\cancel{p = (4x^3p - p)p}$$

$$0 = (4x^3p^2 - 2p) + \left(2x^4p\frac{dp}{dx} - x\frac{dp}{dx}\right)$$

$$0 = 2p[2x^3p - 1] + [2x^3p - 1]x\frac{dp}{dx}$$

$$0 = (2x^3p - 1)\left(2p + x\frac{dp}{dx}\right)$$

$(2x^3p - 1)$ gives us only singular sol^n.

we will take,

$$2p + x\frac{dp}{dx} = 0$$

$$2p = -x\frac{dp}{dx}$$

$$2\frac{dx}{x} = -\frac{dp}{p} \Rightarrow 2\int\frac{dx}{x} = -\int\frac{dp}{p}$$

$$\Rightarrow 2\log x + \log p = c$$

$$\cancel{\log\frac{x^2p}{p} = c}$$

x^2p

We will find the value of. p :-

$\log x^2 p = c$

$x^2 p = c$ then $\boxed{p = c/x^2}$

By Putting in the eqn ① :-

$$y + \frac{c}{x^2} \times x = x^4 \times \frac{c^2}{x^4}$$

$$y + \frac{c}{x} = c^2$$

$$xy + c = xc^2$$

Que → $x^2 \left(\frac{dy}{dx}\right)^4 + 2x \frac{dy}{dx} - y = 0$ ....

Put $\frac{dy}{dx} = p$

$$x^2 p^4 + 2xp - y = 0 \quad \text{----} ①$$

$$y = x^2 p^4 + 2xp$$

diffn w.r.t. x :-

$$\frac{dy}{dx} = x^2 \cdot 4p^3 \frac{dp}{dx} + 2xp^4 + 2\left[x \frac{dp}{dx} + p\right]$$

$$p = 4p^3 x^2 \frac{dp}{dx} + 2xp^4 + 2x \frac{dp}{dx} + 2p$$

$$0 = 4p^3 x^2 \frac{dp}{dx} + 2xp^4 + p + 2x \frac{dp}{dx}$$

$$0 = p\left[2xp^3 + 1\right] + \left[2xp^3 + 1\right] 2x \frac{dp}{dx}$$

$$0 = \left(p + 2x \frac{dp}{dx}\right)\left(2xp^3 + 1\right)$$

This gives only singular solⁿ

$$p + 2x \frac{dp}{dx} = 0$$

$$p = -2x \frac{dp}{dx} \Rightarrow \frac{dx}{x} = -2 \frac{dp}{p}$$

On integration;

$$\int \frac{dx}{x} = -2\int \frac{dp}{p} + c$$

$$\log x = -2\log p + c$$

$-\frac{1}{2}\log x = \log p + \log c$

$$\log x + \log p^2 = c$$

$\log(k\sqrt{x}) = \log p + \log c$

$\boxed{p = c/\sqrt{x}}$

$$\log xp^2 = c \Rightarrow xp^2 = c$$

i.e. $p = \sqrt{c/x}$

By putting the value of p in que sol^n

$$x^2\left(\sqrt{\frac{c}{x}}\right)^4 + 2x\left(\sqrt{\frac{c}{x}}\right) - y = 0$$

$$y = \frac{x^2 \times c^2}{x^2} + 2x\sqrt{\frac{c}{x}}$$

$$y = c^2 + 2x\sqrt{\frac{c}{x}}$$

Ans

Que → $xp^2 + x = 2yp$

$$y = \frac{1}{2p}\left[xp^2 + x\right] = \frac{1}{2}xp + \frac{1}{2}\frac{x}{p}$$

$$\frac{dy}{dx} = \frac{1}{2}\left[x\frac{dp}{dx} + p\right] + \frac{1}{2}\left[-\frac{x}{p^2}\frac{dp}{dx} + \frac{1}{p}\right]$$

$$p = \frac{1}{2}\left(x\frac{dp}{dx} - \frac{x}{p^2}\frac{dp}{dx}\right) + \frac{1}{2}\left(p + \frac{1}{p}\right)$$

$$0 = \frac{1}{2}\left(x\frac{dp}{dx} - \frac{x}{p^2}\frac{dp}{dx}\right) + \frac{1}{2}\left(-\frac{p^2}{2}\right.$$

$$0 = \frac{1}{2}x\frac{dp}{dx}\left[1 - \frac{1}{p^2}\right] + \frac{1}{2}p^2\left(1 - \frac{1}{p}\right)$$

Que :— $xp^2 + x = 2yp$

Sol^n :— $y = \dfrac{xp^2+x}{2p} \Rightarrow \dfrac{dy}{dx} = \dfrac{2p\left(p^2+2px\frac{dp}{dx}+1\right) - 2\frac{dp}{dx}(xp^2+x)}{4p^2}$

$$4p^3 = 2p^3 + 4p^2x\frac{dp}{dx} + 2p - 2xp^2\frac{dp}{dx} - 2x\frac{dp}{dx}$$

$$4p^3 = 2p^3 + 2p + (4p^2x - 2xp^2 - 2x)\frac{dp}{dx}$$

$$2p^3 - 2p = 4p^2x - 2(xp^2+x)\frac{dp}{dx}$$

$$2p(p^2-1) = (2p^2x - 2x)\frac{dp}{dx}$$

$$2p(p^2-1) = 2x(p^2-1)\frac{dp}{dx}$$

$$2x(p^2-1)\frac{dp}{dx} - 2p(p^2-1) = 0$$

$$\left(2x\frac{dp}{dx} - 2p\right)(p^2-1) = 0$$

$$\cancel{2}x\frac{dp}{dx} = \cancel{2}p \Rightarrow x\frac{dp}{dx} = p$$

$$\int\frac{dp}{p} = \int\frac{dx}{x}$$

$$\log p - \log x = \log c \Rightarrow \underline{p = cx}$$

then $y = \dfrac{xc^2x^2 + x}{2cx}$

$$2yc = c^2x^2 + 1$$

Solvable for p

Linear

$y = f(x, p)$

$y = xp^2 + p$

diff $\frac{dy}{dx} = 2xp\frac{dp}{dx} + p^2 + \frac{dp}{dx}$

$\frac{dy}{dx} = (2xp+1)\frac{dp}{dx} + p^2$

$p = (2xp+1)\frac{dp}{dx} + p^2$

$0 = (2xp+1)\frac{dp}{dx} + p^2 - p$

$\frac{dx}{dp} = \cancel{\frac{-(p^2-p)}{2xp+1}} \quad \frac{2xp+1}{-p^2+p}$

$\frac{dx}{dp} = \frac{2xp+1}{p(1-p)} \Rightarrow \frac{dx}{dp} - \frac{2x}{1-p} =$

This is linear in p:

$I.F = e^{\int -2/1-p \, dp} = e^{\frac{-2\log(1-p)}{-1}} = e^{\log(1-p)}$

$I.F = (1-p)^2$

By integration we integrate $(1-p)$ and get

Then Sol^n is $\longrightarrow$

$x \cdot I.F. = \int Q \cdot I.f + C$

$x(1-p)^2 = \int \frac{1}{p(1-p)}(1-p)^2 \, dp + c$

$x(1-p)^2 = \int \frac{1-p}{p} \, dp + c$

$x(1-p)^2 = \int \left(\frac{1}{p} - 1\right) dp + c$

$x(1-p)^2 = \log p - p + c$

Then $x = \frac{\log p - p + c}{(1-p)^2}$

Then the Sol^n is $x = \frac{\log p - p + c}{(1-p^2)}$

& $y = \left(\frac{\log p - p + c}{(1-p)^2}\right)p^2 + p$

6 $y = p\sin p + \cos p \quad \text{---} ①$

$$\frac{dy}{dx} = p\cos p\frac{dp}{dx} + \cancel{\sin p\frac{dp}{dx}} - \cancel{\sin p\frac{dp}{dx}}$$

$$\cancel{p} = \cancel{p}\cos p\frac{dp}{dx}$$

$$1 = \cos p\frac{dp}{dx}$$

$$\frac{1}{\cos p} = \frac{dp}{dx}$$

On integration :—

$$\int dx = \int \cos p\, dp + c$$

$$x = \sin p + c$$

$$\Rightarrow p = \sin^{-1}(x+c)$$

Putting the value of p in ①

$$y = \sin^{-1}(x+c) + \cos\sin^{-1}(x+c)$$

---

7 $p - p^2 = (2xp+1)\frac{dp}{dx}$

$$dx = \frac{2xp+1}{p-p^2}\,dp$$

$$\boxed{\frac{dx}{dp} - \frac{2x}{1-p} = \frac{1}{p-p^2}}$$

Lec - 07

01

Solvable for x

Steps :-
1. $x = f(y, p)$
2. $\frac{dx}{dy}$
3. put $\frac{dx}{dy} = \frac{1}{p}$
4. $F(y, p, c) = 0$ is its sol^n

—o—

Que → $p^3 - 4xyp + 8y^2 = 0$

$$x = \frac{p^3 + 8y^2}{4yp}$$

diff^n w.r.t. y :-

$$\frac{dx}{dy} = \frac{d}{dy}\left[\frac{p^2}{4y} + \frac{2y}{p}\right]$$

$$\frac{dx}{dy} = \frac{1}{4}\left(-\frac{p^2}{y^2} + \frac{1}{y}2p\frac{dp}{dy}\right) + 2\left(y\left\{-\frac{1}{p^2}\right\}\frac{dp}{dy} + \frac{1}{p}\right)$$

$$\frac{1}{p} = \frac{1}{2}\frac{p}{y}\frac{dp}{dy} - \frac{1}{4}\frac{p^2}{y^2} + \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy}$$

$$0 = \frac{dp}{dy}\left[\frac{p}{2y} - \frac{2y}{p^2}\right] + \frac{2}{p} - \frac{1}{4}\frac{p^2}{y^2} - \frac{1}{p}$$

$$0 = \left(\frac{p}{2y} - \frac{2y}{p^2}\right)\frac{dp}{dy} + \frac{1}{p} - \frac{1}{4}\frac{p^2}{y^2}$$

$$0 = \left(\frac{p}{2y} - \frac{2y}{p^2}\right)\frac{dp}{dy} - \frac{p}{2y}\left(\frac{p}{2y} - \frac{2y}{p^2}\right)$$

$$0 = \left(\frac{p}{2y} - \frac{2y}{p^2}\right)\left(-\frac{p}{2y} + \frac{dp}{dy}\right) = 0$$

$\left(\frac{p}{2y} - \frac{2y}{p^2}\right)$ gives us singular sol^n

Then we will take

$$\frac{dp}{dy} - \frac{p}{2y} = 0$$

$$\frac{dp}{dy} = \frac{p}{2y}$$

$$2\frac{dp}{p} = \frac{dy}{\cancel{2}y} \Rightarrow \int \frac{dp}{p} = \int \frac{dy}{2y} + c$$

$$\log p^2 = \cancel{\tfrac{1}{2}} \log y + c$$

$$\log p^2 = \log y + c$$

$$\log \frac{p^2}{y} = c \Rightarrow \frac{p^2}{y} = c$$

then $\underline{p^2 = yc}$

We can write ;

$$p^3 - 4xyp + 8y^2 = 0$$

$$p(p^2 - 4xy) = -8y^2$$

squaring both sides :—

$$p^2(p^2 - 4xy)^2 = 64y^4$$

put $p^2 = yc$

$$yc(yc - 4xy)^2 = 64y^4$$

$$yc[y^2c^2 - 8xy^2c + 16x^2y^2] = 64y^4$$

$$c[y^2c^2 - 8xy^2c + 16x^2y^2] = 64y^3$$

$$cy^2[c^2 - 8xc + 16x^2] = 64y^3$$

$$c[c^2 - 8xc + 16x^2] = 64y$$

$$p^3y^2 + 2px = y$$

$$x = \frac{y - p^3y^2}{2p}$$

$$\frac{dx}{dy} = \frac{d}{dy}\left[\frac{y}{2p} - \frac{p^2y^2}{2}\right]$$

$$\frac{dx}{dy} = \frac{1}{2}\left(y \cdot \frac{-1}{p^2}\frac{dp}{dy} + \frac{1}{p}\right) - \frac{1}{\cancel{2}}\left(p^2 \cdot \cancel{2}y + y^2 \cdot \cancel{2}p\frac{dp}{dy}\right)$$

$$\frac{1}{p} = -\frac{y}{2p^2}\frac{dp}{dy} + \frac{1}{2p} - yp^2 - y^2p\frac{dp}{dy}$$

$$\left(\frac{-y}{2p^2} - y^2p\right)\frac{dp}{dy} + \frac{1}{2p} - yp^2 - \frac{1}{p} = 0$$

$$\left(\frac{-y}{2p^2} - y^2p\right)\frac{dp}{dy} - \frac{1}{2p} - yp^2 = 0$$

$$-y\left[\frac{1}{2p^2} + yp\right]\frac{dp}{dy} - p\left[\frac{1}{2p^2} + yp\right] = 0$$

$$\left(\frac{1}{2p^2} + yp\right)\left[-y\frac{dp}{dy} - p\right] = 0$$

$$-y\frac{dp}{dy} - p = 0$$

$$-y\frac{dp}{dy} = p$$

$$\int\frac{dp}{p} = \int\frac{-dy}{y}$$

$$\log p = -\log y + \log c$$

$$\boxed{p = \frac{c}{y}}$$

$$\frac{c^3}{y^{\cancel{3}}} \times \cancel{y^2} + 2 \times \frac{c}{y}x = y$$

$$\frac{c^3}{y} + \frac{2cx}{y} = y$$

$$c^3 + 2cx = y^2$$

$\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{1}{2}\cosh^{-1}\frac{x}{a}$

Linear

Que. $x - yp = ap^2$

$x = ap^2 + yp$

$\frac{dx}{dy} = a\,2p\frac{dp}{dy} + \left(y\frac{dp}{dy} + p\right)$

$\frac{1}{p} = 2ap\frac{dp}{dy} + y\frac{dp}{dy} + p$

$(2ap+y)\frac{dp}{dy} + p - \frac{1}{p} = 0$

mistake

$(2ap+y)\frac{dp}{dy} = \frac{1}{p} - p$

$\frac{dy}{dp} = \frac{(2ap+y)p}{1-p^2}$

$\frac{dy}{dp} = \frac{p}{1-p^2}y + \frac{2ap^2}{1-p^2}$

$\frac{dy}{dp} - \frac{p}{1-p^2}y = \frac{2ap^2}{1-p^2}$

$\frac{dy}{dp} + \frac{p}{p^2-1}y = \frac{2ap^2}{1-p^2}$

I.F. $\to e^{\int\frac{p}{p^2-1}dp} = e^{\frac{1}{2}\log(p^2-1)} = e^{\log\sqrt{p^2-1}}$

$= \sqrt{p^2-1}$

Correct

Sol^n $y.\sqrt{p^2-1} = \int \frac{2ap^2}{1-p^2}.\sqrt{p^2-1}\,dp + C$

$y\sqrt{p^2-1} = -2a\int\frac{p^2-1+1}{p^2-1}\sqrt{p^2-1}\,dp + C$

$y.\sqrt{p^2-1} = -2a\int\frac{(p^2-1)\sqrt{p^2-1}}{\cancel{p^2-1}}dp - \frac{2a}{p^2-1}\sqrt{p^2-1}\,dp + C$

$y.\sqrt{p^2-1} = -2a.\int\sqrt{p^2-1}\,dp - 2a\int\frac{1}{\sqrt{p^2-1}}dp + C$

$y.\sqrt{p^2-1} = -2a\left\{\frac{p\sqrt{p^2-1}}{2} - \frac{1}{2}\cosh^{-1}p\right\} -$

$2a.\cosh^{-1}p + C$

$y.\sqrt{p^2-1} = -2a\left\{\frac{p}{2}\sqrt{p^2-1}\right\} - \frac{2a}{2}\cosh^{-1}p - 2a\cosh^{-1}p + C$

$y.\sqrt{p^2-1} = -ap\sqrt{p^2-1} - a\cosh^{-1}p + C$

$$y\sqrt{p^2-1} + pa\sqrt{p^2-1} + a\cosh^{-1}p = c$$

$$(y+ap)\sqrt{p^2-1} + a\cosh^{-1}p = c$$

we can solve

By Solvable for y

$$x - yp = ap^2$$

$$yp = x - ap^2$$

$$y = \frac{x-ap^2}{p}$$

$$y = \frac{x}{p} - ap$$

$$\frac{dy}{dx} = \left(x\left(\frac{-1}{p^2}\right)\frac{dp}{dx} + \frac{1}{p}\right) - a\frac{dp}{dx}$$

$$p = -\frac{x}{p^2}\frac{dp}{dx} + \frac{1}{p} - a\frac{dp}{dx}$$

$$0 = \left(-\frac{x}{p^2} - a\right)\frac{dp}{dx} + \frac{1}{p} - p$$

$$\left(\frac{x}{p^2} + a\right)\frac{dp}{dx} = \frac{1-p^2}{p}$$

$$\frac{dx}{dp} = \frac{x+ap^2}{p^2} \times \frac{p}{1-p^2}$$

$$\frac{dx}{dp} = \frac{x+ap^2}{p(1-p^2)}$$

$$\frac{dx}{dp} = \frac{1}{p-p^3}x + \frac{ap}{1-p^2}$$

$$\frac{dx}{dp} + \frac{x}{p^3-p} = \frac{ap}{1-p^2}$$

I.F $\rightarrow$ $e^{\int \frac{1}{p^3-p}dp} = ?$

$\int \frac{1}{p^3-p}dp \rightarrow$

For This $\rightarrow$

$$\frac{1}{p(p^2-1)} = \frac{A}{p} + \frac{Bp+C}{p^2-1}$$

$$1 = Ap^2 - A + Bp^2 + pC$$

$$1 = (A+B)p^2 - A + pC$$

Then $A+B=0$, $-A=1$, $C=0$

i.e. $A = -1$,

$-1+B=0 \Rightarrow \boxed{B=1}$

$$\frac{1}{p(p^2-1)} = \frac{-1}{p} + \frac{p}{p^2-1}$$

Now on integration; $-\log p + \frac{1}{2}\log(p^2-1)$

$$\Rightarrow \log\frac{\sqrt{p^2-1}}{p}$$

Then I.F. is $e^{\log\frac{\sqrt{p^2-1}}{p}} = \frac{\sqrt{p^2-1}}{p}$

$$x \cdot \frac{\sqrt{p^2-1}}{p} = +\int \frac{a\cancel{p}}{-1+p^2}\frac{\sqrt{p^2-1}}{\cancel{p}}\,dp + c$$

$$x \cdot \frac{\sqrt{p^2-1}}{p} = \int -\frac{a}{\cancel{\sqrt{p^2-1}}}\frac{\cancel{\sqrt{p^2-1}}}{\sqrt{p^2-1}}\,dp + c$$

$$\frac{x}{p}\sqrt{p^2-1} = -a\int\frac{1}{\sqrt{p^2-1}}\,dp + c$$

$$\frac{x}{p}\sqrt{p^2-1} = -a\log\left[p+\sqrt{p^2+a^2}\right] + c$$

$$x = \frac{-ap}{\sqrt{p^2-1}}\log\left[p+\sqrt{p^2+a^2}\right] + c \quad \text{---(1)}$$

$$y = \frac{\frac{-ap}{\sqrt{p^2-1}}\log\left[p+\sqrt{p^2+a^2}\right] - ap^2}{p} \quad \text{---(2)}$$

$$p = \tan\left[x - \frac{p}{1+p^2}\right]$$

$$x = \frac{p}{1+p^2} + \tan^{-1} p$$

Diffⁿ w.r.t y:-

$$\frac{dx}{dy} = \frac{(1+p^2)\,dp/dy - p\,2p\,.\,dp/dy}{(1+p^2)^2} + \frac{1}{1+p^2}\cdot\frac{dp}{dy}$$

$$\frac{1}{p} = \frac{dp}{dy}\,\frac{(1-p^2)}{(1+p^2)^2} + \frac{1}{1+p^2}\,\frac{dp}{dy}$$

$$\frac{1}{p} = \frac{dp}{dy}\left[\frac{(1-p^2)+1+p^2}{(1+p^2)^2}\right]$$

$$\frac{1}{p} = \frac{dp}{dy}\cdot\frac{2}{(1+p^2)^2}$$

$$dy = \frac{2p}{(1+p^2)^2}\,dp$$

integration:-

$$y = -\frac{1}{1+p^2} + c$$

$$y + (1+p^2) = c$$

Solvable for p

Que :- $p^2 + 2py\cot x = y^2$

By Addition and subtraction of $y^2\cot^2 x$

$$(p^2 + 2py\cot x + y^2\cot^2 x) = y^2 + y^2\cot^2 x$$

$$(p + y\cot x)^2 = y^2[1+\cot^2 x] = y^2(\operatorname{cosec}^2 x)$$

then $p + y\cot x = \pm y\, \cancel{(1+\cot^2 x)}\, \operatorname{cosec} x$

$p + y\cot x = y\operatorname{cosec} x$, $p + y\cot x = -y\operatorname{cosec} x$

$$\frac{dy}{dx} + y\cot x = y\operatorname{cosec} x$$

$$dy + y\cot x\,dx = y\operatorname{cosec} x\,dx$$

$$dy = y(\operatorname{cosec} x - \cot x)\,dx$$

$$\int\frac{dy}{y} = \int(\operatorname{cosec} x - \cot x)\,dx$$

$$\log y = \log\tan\frac{x}{2} - \log\sin x + \log c$$

$$\log y = \log\left\{\frac{c.\tan x/2}{\sin x}\right\}$$

$$y = \frac{c\tan x/2}{\sin x}$$

$$= \frac{c.\tan x/2}{2\sin x/2.\cos x/2}$$

$$= \frac{c.\frac{\cancel{\sin x/2}}{\cos x/2}}{2\cancel{\sin x/2}.\cos x/2}$$

$$= \frac{c}{2\cos^2 x/2} = \frac{c}{1+\cos x}$$

$$y(1+\cos x) = c$$

Similarly,

$$y = \frac{c}{1-\cos x}$$

$$y(1-\cos x) = c$$

Hence the Solⁿ is

$$y(1 \pm \cos x) = c$$

Ans

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Linear Differential Equation of Order one :—

An Equation in which dependent variable and Its derivative is of degree one and not multiplied Together, is called L.D.E of order one.

$$\boxed{\frac{dy}{dx} + Py = Q}$$

Linear Differential equation of Higher order :—

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + P_3 \frac{d^{n-3}y}{dx^{n-3}} + \cdots\cdots + P_n y = X$$

Where, $P_1, P_2, P_3 \cdots\cdots P_n$ & X are function of x.

$$\boxed{\text{Electro - Mechanical Vibration} \longrightarrow \text{USE}}$$

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Complementary Function

Rules for C.F. $\rightarrow$

(1) change $\frac{dy}{dx} \rightarrow Dy$ which is Diffⁿ operator.
(2) Find Auxiliary Eqⁿ i.e. $f(D) = 0$
(3) Replace $D \rightarrow m$
(4) Find roots $m_1, m_2, m_3 \ldots\ldots$
(5) Now C.F. depends upon roots.

Case (I) When Roots are real and different $\rightarrow$

C.F. $\rightarrow$ $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots$

Case (II) When Roots are real and same $\rightarrow$ let $m_1 = m_2$

C.F. $\rightarrow$ $y = (c_1 + c_2 x) e^{m_1 x}$

Let $m_1 = m_2 + m_3$

$y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x}$

Case (III) When Roots are img and different $\rightarrow$ *[Pair of roots]*

$m_1 = \alpha \pm i\beta$

C.F. $\rightarrow$ $y = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$

Case IV When roots are img and same $\rightarrow$

$m_1 = m_2 = \alpha \pm i\beta$

C.F. $\rightarrow$ $y = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]$

Imp Que ① $D^4y + m^4y = 0$ Imp Que ② $D^4y + 4y = 0$

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Que ① $D^3y - 8y = 0$

$a^3 - b^3 = 0$
$(a-b)(a^2+ab+b^2)$

A.E. $\rightarrow$ $m^3 - 8 = 0$

$(m-2)(m^2+2m+4) = 0$

$m = 2$, $m = \frac{-2 \pm \sqrt{4-16}}{2}$

$= \frac{-2 \pm 2\sqrt{3}i}{2}$

$= -1 \pm \sqrt{3}i$

C.F. $\rightarrow$ $y = c_1e^{2x} + e^{-x}[c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x]$ Ans

Que ② $(D^3 - 6D^2 + 11D - 6)y = 0$

A.E. $\rightarrow$ $m^3 - 6m^2 + 11m - 6 = 0$

$m^2(m-2) - 4m(m-2) + 3(m-2) = 0$

$(m-2)(m^2 - 4m + 3) = 0$

$m = 2$, $m = 1$, $m = 3$

C.F. $\rightarrow$ $y = c_1e^x + c_2e^{2x} + c_3e^{3x}$ Ans

Que ③ $D^2y + 6Dy + 9y = 0$

A.E. $\rightarrow$ $m^2 + 6m + 9 = 0$

$m = -3, -3$

C.F. $\rightarrow$ $y = (c_1 + c_2x)e^{-3x}$ Ans

Que ④ $y'' + 2y' + 10y = 0$

$y(0) = 4$ & $y'(0) = 1$

$\Rightarrow D^2y + 2Dy + 10y = 0$

A.E. $\rightarrow$ $m^2 + 2m + 10 = 0$

$m = -1 \pm 3i$

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C.F. $\rightarrow$ $y = e^{-x}\left[c_1 \cos 3x + c_2 \sin 3x\right]$

By Applying Initial condition;

put $y = 4$ at $x = 0$

$\boxed{c_1 = 4}$

Again,

$$\frac{dy}{dx} = e^{x}\left[-3c_1 \sin 3x + 3c_2 \cos 3x\right] + (-e^{x})\left[c_1 \cos 3x + c_2 \sin 3x\right]$$

Put, $y = 1$ at $x = 0$

$1 = 3c_2 - 4$

$\therefore \boxed{c_2 = 5/3}$

Then $y = e^{-x}\left[4 \cos 3x + \frac{5}{3} \sin 3x\right]$

Ans.

Que (5) $D^4 y - m^4 y = 0$

Then Prove that

$y = c_1 \cos mx + c_2 \sin mx + c_3 \cosh mx + c_4 \sinh mx$

A.E. $\rightarrow$ $D^4 - m^4 = 0$

$(D^2)^2 - (m^2)^2 = 0$

$(D^2 - m^2)(D^2 + m^2) = 0$

$D = \pm m$, $D = \pm mi$

C.F. $\rightarrow$ $y = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx$ — ①

Now we will prove that

$y = c_1 \cos mx + c_2 \sin mx + c_3 \cosh mx + c_4 \sinh mx$

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By ① →

Let $C_1 = \frac{A+B}{2}$, $C_2 = \frac{A-B}{2}$

$$y = \left(\frac{A+B}{2}\right) e^{mx} + \left(\frac{A-B}{2}\right) e^{-mx} + C_3 \cos mx + C_4 \sin mx$$

$$y = \frac{1}{2}\left[Ae^{mx} + Ae^{-mx}\right] + \frac{1}{2}\left[Be^{mx} + Be^{-mx}\right] + C_3 \cos mx + C_4 \sin mx$$

$$y = A\left[\frac{e^{mx} + e^{-mx}}{2}\right] + B\left[\frac{e^{mx} - e^{-mx}}{2}\right] + C_3 \cos mx + C_4 \sin mx$$

$$y = A \cosh mx + B \sinh mx + C_3 \cos mx + C_4 \sin mx$$

Again $A = C_1$ & $B = C_2$

$$y = C_1 \cosh mx + C_2 \sinh mx + C_3 \cos mx + C_4 \sin mx$$

Ans

Que ⑥ $D^3y + 2D^2y + Dy = 0$

A.E. $m^3 + 2m^2 + m = 0$

$m = 0, -1, -1$

Then C.F. → $y = C_1 + (C_2 + C_3 x)e^{-x}$

Ans

Que ⑦ $(D^4 + 8D^2 + 16)y = 0$

$(D^2+4)^2 = 0 \Rightarrow D = \pm 2i, \pm 2i$

$y = [(C_1 + C_2 x)\cos 2x + (C_3 + C_4 x)\sin 2x]$

Particular Integral

$\frac{1}{f(D)} X$ is that function of x, not containing arbitary Constant.

Case I → $\boxed{X = e^{ax}}$

Replacing $D \longrightarrow a$

since, $D(e^{ax}) = a e^{ax}$, $D^2(e^{ax}) = a^2 e^{ax}$, $D^4(e^{ax}) = a^4 e^{ax}$ {By comparing $D \to a$}

i.e. $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$

Provided $f(a) \neq 0$

Que ① $D^3y + 2D^2y + Dy = e^{2x}$

P.I. $\frac{1}{D^3 + 2D^2 + D} e^{2x}$ $\quad D = a = 2$

$\frac{1}{8 + 8 + 2} e^{2x}$

$\frac{e^{2x}}{18}$ Ans.

Que ② $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$

$\frac{1}{D^3 - 6D^2 + 11D - 6} e^{-2x} + e^{-3x}$

$\frac{1}{D^3 - 6D^2 + 11D - 6} e^{-2x} + \frac{1}{D^3 - 6D^2 + 11D - 6} e^{-3x}$

$\frac{e^{-2x}}{-60} + \frac{e^{-3x}}{-120}$ Ans

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Que ③ $D^2y + 4Dy + 5y = -2\cosh x$

$$\frac{1}{D^2+4D+5} - 2\cosh x$$

$$\frac{-2}{D^2+4D+5}\left(\frac{e^x+e^{-x}}{2}\right)$$

$$\frac{-1}{D^2+4D+5}e^x - \frac{1}{D^2+4D+5}e^{-x}$$

$$= \frac{-e^x}{10} - \frac{e^{-x}}{2}$$

Then, $-\frac{1}{2}\left[\frac{e^x}{5} + e^{-x}\right]$ Ans

Que ④ $(D^2 + 2D + 1)y = e^{2x}$

$$\frac{1}{D^2+2D+1}e^{2x}$$

$$\frac{e^{2x}}{9}$$ Ans

Que :- Solve, $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = e^{2x}\sin 2x$

Hence, Prove that $y = e^{2x}[A\cosh\sqrt{3}x + B\sinh\sqrt{3}x]$

Solve :- $(D^2 - 4D + 1)y = 0$

$m = 2 \pm \sqrt{3}$ (Real)

C.F.

$y = c_1e^{(2+\sqrt{3})x} + c_2e^{(2-\sqrt{3})x}$

let $c_1 = \frac{A+B}{2}$, $c_2 = \frac{A-B}{2}$

$y = \left(\frac{A+B}{2}\right)e^{(2+\sqrt{3})x} + \left(\frac{A-B}{2}\right)e^{(2-\sqrt{3})x}$

$= \frac{1}{2}\left[Ae^{(2+\sqrt{3})x} + Ae^{(2-\sqrt{3})x}\right] +$

$\frac{1}{2}\left[Be^{(2+\sqrt{3})x} - Be^{(2-\sqrt{3})x}\right]$

$= \frac{1}{2}e^{2x}\left[A\left(e^{\sqrt{3}x} + e^{-\sqrt{3}x}\right)\right] +$

$\frac{1}{2}e^{2x}\left[B\left(e^{\sqrt{3}x} - e^{-\sqrt{3}x}\right)\right]$

$= e^{2x}\left[A\cosh\sqrt{3}x + B\sin\sqrt{3}x\right]$

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Case (II)

$$\boxed{X = \sin(ax+b) \text{ OR } \cos(ax+b)}$$

Replacing, $D^2 \longrightarrow -a^2$

since,

$D[\sin(ax+b)] = a\cos(ax+b)$

$D^2[\sin(ax+b)] = -a^2 \sin(ax+b)$

Function does not change

If we put $D^2 \longrightarrow -a^2$

Que (1) $D^2x + 2Dx + 3x = \sin t$

$\frac{1}{D^2+2D+3} \sin t$ $\qquad D^2 = -a^2 = -1$

$\Rightarrow \frac{1}{-1+2D+3} \sin t$

$\Rightarrow \frac{1}{2+2D} \sin t$

$\Rightarrow \frac{1}{2D+2} \times \frac{2D-2}{2D-2} \sin t$

$\Rightarrow \frac{(2D-2)\sin t}{4D^2-4}$ $\qquad D^2 = -a^2 = -1$

$\Rightarrow \frac{(2D-2)}{-4-4} \sin t$

$\Rightarrow -\frac{2}{8} D(\sin t) + \frac{1}{4} \sin t$

$\Rightarrow -\frac{1}{4} \cos t + \frac{1}{4} \sin t$

$y = \frac{\sin t}{4} - \frac{\cos t}{4}$

Ans

Que ② $(D^2+3D+2)y = 4\cos^2 x$

$$\frac{1}{D^2+3D+2}\,4\cos^2 x$$

$$\frac{4}{D^2+3D+2}\left[\frac{1+\cos 2x}{2}\right]$$

$$\frac{2}{D^2+3D+2}\cos 2x + \frac{2}{D^2+3D+2}$$

$$\frac{2}{-4+3D+2}\cos 2x + \frac{2}{D^2+3D+2}e^{0x}$$

$$\Rightarrow \frac{2\cos 2x}{-2+3D} + \frac{2}{2}$$

$$\Rightarrow \frac{2}{3D-2}\times\frac{3D+2}{3D+2}\cos 2x + 1$$

$$\Rightarrow \frac{2(3D+2)}{9D^2-4}\cos 2x + 1$$

$$\Rightarrow \frac{-6\times 2\sin 2x + 4\cos 2x}{-36-4} + 1$$

$$\Rightarrow \frac{-12\sin 2x + 4\cos 2x}{-40} + 1$$

$$\Rightarrow \frac{12}{40}\sin 2x - \frac{1}{10}\cos 2x + 1$$

$$\Rightarrow \frac{3}{10}\sin 2x - \frac{1}{10}\cos 2x + 1$$

Que ③ $D^3y + 2D^2y + Dy = e^{2x} + \sin 2x$

$$\frac{1}{D^3+2D^2+D}\,e^{2x} + \sin 2x$$

$$\frac{1}{D^3+2D^2+D}e^{2x} + \frac{1}{D^3+2D^2+D}\sin 2x$$

$$\frac{e^{2x}}{18} + \frac{1}{D(-4)+2\times(-4)+D}\sin 2x$$

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$$\frac{e^{2x}}{18} + \frac{1}{-3D-8}\sin 2x$$

$$\frac{e^{2x}}{18} - \left[\frac{1}{3D+8} \times \frac{3D-8}{3D-8}\right]\sin 2x$$

$$\frac{e^{2x}}{18} - \left[\frac{3D-8}{9D^2-64}\right]\sin 2x$$

$$\frac{e^{2x}}{18} - \frac{3D-8}{-36-64}\sin 2x$$

$$\frac{e^{2x}}{18} + \frac{3D-8}{100}\sin 2x$$

$$\frac{e^{2x}}{18} + \frac{3\times 2\cos 2x}{100} - \frac{8}{100}\sin 2x$$

$$\frac{e^{2x}}{18} + \frac{3}{50}\cos 2x - \frac{2}{25}\sin 2x$$

Ans.

Que $(D^2 + 2D + 1)y = e^{2x} - \cos^2 x$

$$\frac{1}{D^2+2D+1} e^{2x} - \cos^2 x$$

Ans $\rightarrow$ $\frac{e^{2x}}{9} - \frac{2}{25}\sin 2x + \frac{3}{50}\cos 2x$

$-1/2$ Ans.

Que $(D^2+1)y = \cos x$

$\uparrow$
(F.C.)

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Case III   When $X = x^m$

$$\frac{1}{f(D)} x^m$$

$$\Rightarrow \left[f(D)\right]^{-1} x^m$$

Expanding $f(D)$ by using Binomial Theorem.
and multiplying and operate it.

Que ① $(D^2+4)y = x^2$

$$\frac{1}{D^2+4} x^2$$

$$\Rightarrow \frac{1}{4}\left[1+\frac{D^2}{4}\right]^{-1} x^2$$

$$\Rightarrow \frac{1}{4}\left[1-\frac{D^2}{4}+\left(\frac{D^2}{4}\right)^2+ \cdots\cdots\right]x^2$$

$$\Rightarrow \frac{1}{4}\left[x^2-\frac{D^2}{4}x^2\right]$$

$$\Rightarrow \frac{x^2}{4}-\frac{1}{8}$$ Ans

Que ② $D^2y + Dy = x^2+2x+4$

$$\frac{1}{D^2+D} x^2+2x+4$$

$$\frac{1}{D(D+1)}(x^2+2x+4)$$

$$\frac{1}{D}(1+D)^{-1}(x^2+2x+4)$$

$$\frac{1}{D}\left[1-D+\frac{(-1)(-2)}{2!}D^2+\cdots\right](x^2+2x+4)$$

$$\frac{1}{D}(x^2+2x+4-2x-2+2)$$

$$\frac{x^3}{3}+4x$$ Ans.

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Case (IV) When $x = e^{ax} \cdot v$

Replace, $D \rightarrow D+a$

Que (1) $(D^2 - 2D + 4)y = e^x \cdot \cos x$

$$\frac{1}{D^2 - 2D + 4} e^x \cos x$$

$D \rightarrow D+1$

$$\frac{e^x}{(D+1)^2 - 2(D+1) + 4} \cos x$$

$$\frac{e^x}{D^2 + \cancel{2D} + 1 - \cancel{2D} - 2 + 4} \cos x$$

$$\frac{e^x}{D^2 + 3} \cos x$$

$D^2 = -a^2 = -1$

$$\frac{e^x}{-1+3} \cos x$$

$$\frac{e^x}{2} \cos x$$

Que (2)

$(D^4 - 1)y = e^x \cos x$

$$\frac{1}{D^4 - 1} e^x \cos x$$

$$\frac{e^x}{(D+1)^4 - 1} \cos x$$

$$\frac{e^x}{[(D+1)^2]^2 - 1^2} \cos x$$

$$\frac{e^x}{[(D+1)^2+1][(D+1)^2-1]} \cos x$$

$\boxed{OR}$

$$\frac{1}{D^4-1} e^x \cos x$$

$$\frac{1}{(D^2)^2 - 1^2} e^x \cos x$$

$$\frac{1}{(D^2-1)(D^2+1)} e^x \cos x$$

$$\frac{e^x}{[(D+1)^2-1][(D+1)^2+1]} \cos x$$

$$\frac{e^x}{(D^2+2D)(D^2+2D+2)}\cos x$$

$$\frac{e^x}{(\cancel{D^2}_{-1}+2D)(-1+2D+2)}\cos x$$

$$\frac{e^x}{(2D-1)(2D+1)}\cos x$$

$$\frac{e^x}{4D^2-1}\cos x \qquad D^2=-a^2=-1$$

$$-\frac{e^x\cos x}{5}$$

Ans.

Que (3) $\rightarrow$

$$(D^2+4)y = e^x\sin^2 x$$

$$\frac{1}{D^2+4}e^x\sin^2 x$$

$$\frac{1}{D^2+4}e^x\left(\frac{1-\cos 2x}{2}\right)$$

$$\Rightarrow \frac{1}{2}\cdot\frac{1}{D^2+4}e^x - \frac{1}{2}\frac{1}{D^2+4}e^x\cos 2x$$

$$\Rightarrow \frac{1}{2}\,\frac{1}{5}e^x - \frac{1}{2}\,\frac{e^x}{(D+1)^2+4}\cos 2x$$

$$\Rightarrow \frac{e^x}{10} - \frac{1}{2}\,\frac{e^x}{D^2+2D+5}\cos 2x$$

$$\Rightarrow \frac{e^x}{10} - \frac{1}{2}\,\frac{e^x(1-2D)\cos 2x}{1-4D^2}$$

$$\Rightarrow \frac{e^x}{10} - \frac{1}{34}e^x\left[\cos 2x + 4\sin 2x\right]$$

Ans.

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Que $\rightarrow$ $(D^2+4D+3)y = e^{-x}\sin x + x$

$$\Rightarrow \frac{1}{D^2+4D+3} e^{-x}\sin x + x$$

$$\Rightarrow \frac{1}{D^2+4D+3} e^{-x}\sin x + \frac{1}{D^2+4D+3} x$$

$$\Rightarrow \frac{e^{-x}}{(D-1)^2+4(D-1)+3}\sin x + \frac{1}{3\left(1+\frac{D^2}{3}+\frac{4}{3}D\right)} x$$

$$\Rightarrow \frac{e^{-x}}{(D^2-2D+1)+(4D-4)+3}\sin x + \frac{1}{3}\left[1+\left(\frac{D^2}{3}+\frac{4D}{3}\right)\right]^{-1} x$$

$$\Rightarrow \frac{e^{-x}}{D^2+2D}\sin x + \frac{1}{3}\left[1-\left(\frac{4}{3}D+\frac{D^2}{3}\right)+\dots\right] x$$

$$\Rightarrow \frac{e^{-x}}{2D-1}\sin x + \frac{1}{3}\left[x-\frac{4}{3}\cdot 1\right]$$

By Rationalize $(2D-1)$

$$\Rightarrow \frac{e^{-x}(1+2D)}{4D^2-1}\sin x + \frac{x}{3} - \frac{4}{9}$$

$$\Rightarrow \frac{e^{-x}\sin x}{-5} + \frac{2}{-5}e^{-x}\cos x + \frac{x}{3} - \frac{4}{9}$$

Ans.

Faluire Case

(1) If $x = e^{ax}$

$$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$$ & $f(a) = 0$

Then

$$\frac{x}{f'(D)} e^{ax}$$

(2) If $x = \sin(ax+b)$ OR $\cos(ax+b)$

$$\frac{1}{f(D)} \sin(ax+b)$$

$$\frac{1}{f(-a^2)} \sin(ax+b)$$ and $f(-a^2) \neq 0$

$$\frac{x}{f'(-a^2)} \sin(ax+b)$$ if $f'(-a^2) = 0$

Then

$$\frac{x^2}{f''(-a^2)} \sin(ax+b)$$

General Case

When x is another function of x.

Then P.I. $\frac{1}{f(D)} \cdot X$

Resolving $f(D)$ into Partial fraction

let $f(D) = (D-m_1)(D-m_2) \cdots (D-m_n)$

$$\frac{1}{f(D)} = \frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \cdots + \frac{A_n}{D-m_n}$$

$$\text{P.I.} = \left[\frac{A_1}{D-m_1} + \frac{A_2}{D-m_2} + \cdots\right] \cdot X$$

$$= \frac{A_1}{D-m_1} X + \frac{A_2}{D-m_2} X + \cdots$$

$$= A_1 \cdot e^{m_1 x} \int X \cdot e^{-m_1 x}\, dx + A_2 \cdot e^{m_2 x} \int X \cdot e^{-m_2 x}\, dx + \cdots$$

Questions of Faluire Case

Que $\Rightarrow$ $\cancel{(D^2+5D+6)y = e^x}$

(FC) $\left[(D+2)(D-1)^2\right]y = e^{-2x} + 2\sinh x$

Solve $\rightarrow$

$$\frac{1}{(D+2)(D-1)^2}\, e^{-2x} + 2\sinh x$$

$$\frac{1}{(-2+2)(-2-1)^2}\, e^{-2x} + \frac{\cancel{2}}{(D+2)(D-1)^2}\,\frac{e^{x}-e^{-x}}{\cancel{2}}$$

Is Faluire Case

$$\frac{1}{(D+2)(D-1)^2}\, e^{x} - \frac{1}{(D+2)(D-1)^2}\, e^{-x}$$

Faluire case $\qquad$ simple case

Let us ~~we~~ Evaluate each of these terms separately.

(I) $\rightarrow$ $$\frac{x}{\frac{d}{dD}\{D^3-\cancel{2D^2}+D+\cancel{2D^2}-4D+2\}}\, e^{-2x}$$

$$\Rightarrow \frac{x}{\frac{d}{dD}(D^3-3D+2)}\, e^{-2x}$$

$$\Rightarrow \frac{x}{3D^2-3}\, e^{-2x} = \frac{x}{9}\, e^{-2x} \quad \text{------(1)}$$

(II) $\rightarrow$ $$\frac{x}{\frac{d}{dD}\{D^3-\cancel{2D^2}+D+\cancel{2D^2}-4D+2\}}\, e^{x}$$

$$\Rightarrow \frac{x}{3D^2-3}\, e^{x}$$

$$\Rightarrow \frac{x}{3-3}\, e^{x}$$

which is again

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Faluire case ;

$$\Rightarrow \frac{x^2}{\frac{d}{dD}(3D^2-3)} e^x$$

$$\Rightarrow \frac{x^2}{6D} e^x \qquad \text{Now put } D=1$$

$$\Rightarrow \frac{x^2 e^x}{6} \quad \text{-----} (2)$$

(III) $$\frac{1}{(D+2)(D-1)^2} e^{-x}$$

$$\frac{1}{D^3-3D+2} e^{-x}$$

$$\frac{1}{-1+3+2} e^{-x} = \frac{-e^{-x}}{4} \quad \text{-----} (3)$$

D(-1) x

We get, By (1), (2) & (3)

P.I. :- $$y = \frac{x}{9} e^{-2x} + \frac{x^2 e^x}{6} - \frac{e^{-x}}{4}$$

Ans

Que ; $(D^2-4)y = \cosh(2x-1) + 3^x$

(FC) $$\frac{1}{D^2-4} \cosh(2x-1) + \frac{1}{D^2-4} 3^x$$

$$\frac{1}{D^2-4}\left[\frac{e^{2x-1}+e^{-(2x-1)}}{2}\right] + \frac{1}{D^2-4} e^{\log 3^x}$$

$$\frac{1}{D^2-4} \frac{e^{2x}.e^{-1}+e^{-2x}.e^{1}}{2} + \frac{1}{D^2-4} e^{x\log 3}$$

$$\frac{1}{2}\frac{e^{-1}}{D^2-4} e^{2x} + \frac{1}{2}\frac{e}{D^2-4} e^{-2x} + \frac{1}{D^2-4} e^{(\log 3)x}$$

$$\frac{e^{-1}}{2(+4-4)} e^{2x} + \frac{1}{2}\frac{e}{(4-4)} e^{-2x} + \frac{1}{(\log 3)^2-4} e^{(\log 3)x}$$

$\downarrow$ Faluire Case $\quad$ $\downarrow$ Faluire case $\quad$ $\downarrow$ Simple case

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Let, us Evaluate each terms Separately ;

(I) $\frac{1}{2} \frac{x e^{-1}}{\frac{d}{dD}(D^2-4)} e^{2x}$

$\Rightarrow \frac{1}{2e} \frac{x}{2D} e^{2x}$

$\Rightarrow \frac{1}{4e} x \times \frac{e^{2x}}{2} = \frac{x e^{2x}}{8e}$ ---- (1)

(II) $\frac{1}{2} \frac{x e}{\frac{d}{dD}(D^2-4)} e^{-2x}$

$\Rightarrow \frac{1}{2} \frac{xe}{2D} e^{-2x}$

$\Rightarrow \frac{xe}{4} \frac{e^{-2x}}{-2}$

$\Rightarrow -\frac{xe \cdot e^{-2x}}{8}$ ----- (2)

(III) $\frac{1}{(\log 3)^2 - 4} e^{(\log 3)x}$

$\Rightarrow \frac{3^x}{(\log 3)^2 - 4}$ ----- (3)

By (1), (2) and (3)

$$y = \frac{x e^{2x}}{8e} - \frac{xe \cdot e^{-2x}}{8} + \frac{3^x}{(\log 3)^2 - 4}$$

$$= \frac{x e^{2x-1}}{8} - \frac{x}{8} e^{-(2x-1)} + \frac{3^x}{(\log 3)^2 - 4}$$

$$= \frac{x}{4} \sinh(2x-1) + \frac{3^x}{(\log 3)^2 - 4}$$

Ans

Another Questions of P.I.

Que $\Rightarrow$ $D^2y + 2y = x^2e^{3x} + e^x\cos 2x$

$$\Rightarrow \frac{1}{D^2+2}x^2e^{3x} + \frac{1}{D^2+2}e^x\cos 2x$$

$$\Rightarrow \frac{e^{3x}}{(D+3)^2+2}x^2 + \frac{e^x}{(D+1)^2+2}\cos 2x$$

$$\Rightarrow \frac{e^{3x}}{D^2+9+6D+2}x^2 + \frac{e^x}{D^2+2D+1+2}\cos 2x$$

$$\Rightarrow \frac{e^{3x}}{D^2+6D+11}x^2 + \frac{e^x}{D^2+2D+3}\cos 2x$$

$$\Rightarrow \frac{e^{3x}}{11\left[1+\frac{D^2}{11}+\frac{6}{11}D\right]}x^2 + \frac{e^x}{-4+2D+3}\cos 2x \qquad \frac{-1}{D(1)}\,x$$

$$\Rightarrow \frac{e^{3x}}{11}\left[1-\left(\frac{D^2}{11}+\frac{6D}{11}\right)+\left(\frac{D^2}{11}+\frac{6D}{11}\right)^2\right]x^2 + \frac{e^x}{-1+2D}\cos 2x$$

other terms are neglagible

$$\Rightarrow \frac{e^{3x}}{11}\left\{x^2-\frac{2}{11}-\frac{6}{11}2x+\frac{36}{121}\times 2\right\} + \frac{e^x}{-1+2D}\cdot\frac{(-1-2D)}{(-1-2D)}\cos 2x$$

$$\Rightarrow \frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}-\frac{2}{11}+\frac{72}{121}\right) + e^x\frac{(-1-2D)\cos 2x}{4D^2-1}$$

$$\Rightarrow \frac{e^{3x}}{11}\left(x^2-\frac{12}{11}x-\frac{2}{11}+\frac{72}{121}\right) + e^x\frac{(-1-2D)\cos 2x}{-16-1}$$

$$\Rightarrow \frac{e^{3x}}{11}\left(x^2-\frac{12}{11}x+\frac{50}{121}\right) - \frac{e^x}{-17}(-\cos 2x+4\sin 2x)$$

$$\Rightarrow \frac{e^{3x}}{11}\left(x^2-\frac{12}{11}x+\frac{50}{121}\right) + \frac{e^x}{-17}\cos 2x - \frac{4e^x\sin 2x}{-17}$$

$$\Rightarrow \frac{e^{3x}}{11}\left(x^2-\frac{12}{11}x+\frac{50}{121}\right) - \frac{e^x}{17}\cos 2x + \frac{4}{17}e^x\sin 2x$$

$$\Rightarrow \frac{e^{3x}}{11}\left(x^2-\frac{12}{11}x+\frac{50}{121}\right) + \frac{e^x}{17}(4\sin 2x - \cos 2x)$$

Ans

$$= -\frac{x\sin x}{2} - \cos x + \frac{e^x}{2}\left[x + \frac{x}{3} - \frac{x^2}{2} + \frac{x}{2}\right]$$

$$= -\frac{x\sin x}{2} - \cos x + \frac{e^x}{2}\left[\frac{11x}{6} - \frac{x^2}{2}\right]$$

Ans.

General Method *

Std form $\rightarrow$ $\frac{A_1}{D-m_1}\cdot X + \frac{A_2}{D-m_2}X$

Sol^n $\Rightarrow$ $A_1 e^{m_1 x}\int X\cdot e^{-m_1 x}dx + A_2 e^{m_2 x}\int X\cdot e^{-m_2 x}dx$

Que $\rightarrow$ $(D^2+3D+2)y = e^{e^x}$

(split)

$$\Rightarrow \frac{1}{D^2+3D+2}e^{e^x} = \frac{1}{(D+1)(D+2)}e^{e^x}$$

(Solve By Partial Fraction)

$$\Rightarrow \left(\frac{A_1}{D+1} + \frac{A_2}{D+2}\right)e^{e^x} \quad \text{---} (1)$$

$$\Downarrow$$

$$\frac{1}{(D+1)(D+2)} = \frac{A_1}{D+1} + \frac{A_2}{D+2}$$

$$1 = A_1(D+2) + A_2(D+1)$$

$$1 = (A_1+A_2)D + 2A_1 + A_2$$

$2A_1 + A_2 = 1$, $A_1 + A_2 = 0$

$A_1 = -A_2$

$-2A_2 + A_2 = 1$

$\boxed{-A_2 = 1} \Rightarrow \boxed{A_1 = 1}$

$\boxed{\frac{A}{D-m_1}\cdot X}$

By (1) $\longrightarrow$

$$\left(\frac{1}{D+1} - \frac{1}{D+2}\right)e^{e^x}$$

$$\Rightarrow \frac{1}{D+1}e^{e^x} - \frac{1}{D+2}e^{e^x} = \frac{e^{e^x}}{D-(-1)} - \frac{e^{e^x}}{D-(-2)}$$

($m_1 = -1$, $m_2 = -2$)

By using Formula; $\boxed{A_1 e^{m_1 x}\int x e^{-m_1 x}dx + A_2 e^{m_2 x}\int x e^{-m_2 x}dx}$

$$1.\, e^{-x}\int e^{e^x}e^{+x}dx + (-1).\,e^{-2x}\int e^{e^x}e^{+2x}dx$$

$$\Rightarrow e^{-x}\int e^{e^x}.e^x dx - e^{-2x}\int e^{e^x}.e^x.e^x dx$$

Let $e^x = t \Rightarrow e^x dx = dt$

$$\int u.v = uv_1 - u'v_2 + u''v_3 \ldots$$

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$$\Rightarrow e^{-x}\int e^t dt - e^{-2x}\int \underset{v}{e^t} \cdot \underset{u}{t}\, dt$$

$$\Rightarrow e^{-x}\left[e^t\right] - e^{-2x}\left[t.e^t - 1.e^t\right]$$

$$\Rightarrow e^{-x}.e^{e^x} - e^{-2x}\left[e^x.e^{e^x} - e^{e^x}\right]$$

$$\Rightarrow \cancel{e^{-x}.e^{e^x}} - \cancel{e^{-x}.e^{e^x}} + e^{-2x}.e^{e^x}$$

$$\Rightarrow e^{-2x}.e^{e^x}$$

P.I →

$$y = e^{-2x}.e^{e^x}$$

Ans

Que →

$$(D^2+a^2)y = \tan ax$$

$$\Rightarrow \frac{1}{D^2+a^2}\tan ax = \frac{1}{(D+ai)(D-ai)}\tan ax$$

⇒ Solve By Partial fraction;

$$\frac{1}{(D+ai)(D-ai)} = \frac{A}{D+ai} + \frac{B}{D-ai} \quad \text{--- (1)}$$

$$1 = A(D-ai) + B(D+ai)$$

$$1 = (A+B)D - ai(A-B)$$

$A+B=0$ and $-ai(A-B)=1$

$A=-B$ $\qquad -ai(-2B)=1$

$$\boxed{B = 1/2ai}$$

and $\boxed{A = \frac{-1}{2ai}}$

By (1)

$$\frac{1}{2ai}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right]$$

Then P.I. →

$$\frac{1}{2ai}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right]\tan ax$$

$\int \sec x\,dx = \log(\sec x + \tan x)$

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$\frac{1}{D-ai}\tan ax \Rightarrow$ By Using Formula;

$\Rightarrow e^{aix}\int \tan ax \cdot e^{-aix}\,dx$

$\Rightarrow e^{aix}\int \frac{\sin ax}{\cos ax}\left\{\cos ax - i\sin ax\right\}dx$

$\Rightarrow e^{aix}\left[\int \sin ax\,dx - \int \frac{i\sin^2 ax}{\cos ax}\,dx\right]$

$\Rightarrow e^{aix}\left[\frac{-\cos ax}{a} - \int \frac{i(1-\cos^2 ax)}{\cos ax}\,dx\right]$

$\Rightarrow e^{aix}\left[\frac{-\cos ax}{a} - \frac{i}{a}\log(\sec ax + \tan ax) + i\frac{\sin ax}{a}\right]$

$\Rightarrow -\frac{e^{aix}}{a}\left[\cos ax + i\log(\sec ax + \tan ax) - i\sin ax\right]$

$\Rightarrow -\frac{e^{aix}}{a}\left[\underbrace{\cos ax - i\sin ax} + i\log(\sec ax + \tan ax)\right]$

$\Rightarrow -\frac{e^{+aix}}{a}\left[e^{-aix} + i\log(\sec ax + \tan ax)\right]$

$\Rightarrow -\frac{1}{a}\left[1 + ie^{aix}\log(\sec ax + \tan ax)\right]$ —— (1)

In Eq^n (1), we will put $i = -i$

Then we get,

$\frac{1}{D+ai}\tan ax = -\frac{1}{a}\left[1 - ie^{-aix}\log(\sec ax + \tan x\right.$ —— (2)

Now,

$\frac{1}{2ai}\left[-\frac{1}{a}\left\{1 + ie^{aix}\log(\sec ax + \tan ax)\right\} + \frac{1}{a}\left\{1 - ie^{-ax}\log(\sec ax + \tan ax)\right\}\right]$

$\Rightarrow \frac{1}{2ai}\left[-\cancel{\frac{1}{a}} - \frac{1}{a}ie^{aix}\log(\sec ax + \tan ax) + \cancel{\frac{1}{a}} - ie^{-aix}\log(\sec ax + \tan ax)\right.$

$\Rightarrow -\frac{1}{2a^2} e^{aix} \log(\sec ax + \tan ax) -$

$\frac{1}{2a^2} e^{-aix} \log(\sec ax + \tan ax$

$\Rightarrow \frac{-1}{a^2} \log(\sec ax + \tan ax) \left[\frac{e^{aix} + e^{-aix}}{2} \right]$

$\Rightarrow \frac{-1}{a^2} \log(\sec ax + \tan ax) \cdot \cos ax.$

Ans

Que → $(D^3 + 2D^2 + D)y = x^2 e^{2x} + \sin^2 x$

$\frac{1}{D^3 + 2D^2 + D} x^2 e^{2x} + \frac{1}{D^3 + 2D^2 + D} \left\{ \frac{1 - \cos 2x}{2} \right\}$

$\frac{e^{2x}}{(D+2)^3 + 2(D+2)^2 + (D+2)} x^2 + \frac{1}{2} \frac{e^{0x}}{(D^3 + 2D^2 + D)} - \frac{1}{2} \frac{1}{D^3 + 2D^2 + D} \cos 2x$

I   II   III

I $\frac{e^{2x}}{D^3 + 8 + 3D^2 \times 2 + 3D \times 4 + 2(D^2 + 4 + 4D) + D + 2} x^2$

$\Rightarrow \frac{e^{2x}}{D^3 + 8 + 6D^2 + 12D + 2D^2 + 8D + 8 + D + 2} x^2$

$\Rightarrow \frac{e^{2x}}{D^3 + 8D^2 + 21D + 18} x^2$

$\Rightarrow \frac{e^{2x}}{18\left[1 + \frac{D^3}{18} + \frac{8}{18}D^2 + \frac{21}{18}D\right]} x^2$

$\Rightarrow \frac{e^{2x}}{18} \left[1 + \left(\frac{D^3}{18} + \frac{8}{18}D^2 + \frac{21}{18}D\right)\right]^{-1} x^2$

$\Rightarrow \frac{e^{2x}}{18} \left[1 - \left(\frac{D^3}{18} + \frac{8}{18}D^2 + \frac{21}{18}D\right) + (\quad)^2 - \right] x^2$

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$$\Rightarrow \frac{e^{2x}}{18}\left[1+\left(\frac{D^3}{18}+\frac{8}{18}D^2+\frac{21}{18}D\right)\right]^{-1}x^2$$

$$\Rightarrow \frac{e^{2x}}{18}\left[1-\left(\frac{D^3}{18}+\frac{8}{18}D^2+\frac{21}{18}D\right)+\left(\quad\right)^2\right]x^2$$

I $\Rightarrow \frac{e^{2x}}{18}\left[x^2-\frac{8}{18}\times 2 \cancel{x} - \frac{21}{18}\times 2x + \frac{21}{18}\times 2\right]$

$$\Rightarrow \frac{e^{2x}}{18}\left(x^2-\frac{8}{9}-\frac{7}{3}x+\frac{7}{3}\right]$$

III $\Rightarrow \frac{e^{2x}}{18}\left(x^2-\frac{7}{3}x-\frac{8}{9}+\frac{7}{3}\right)$

$$\Rightarrow \frac{e^{2x}}{18}\left[x^2-\frac{7}{3}x+\frac{13}{9}\right] \quad \text{--- (1)}$$

II $\frac{1}{2}\frac{e^{0x}}{D^3+2D^2+D}$

$$\Rightarrow \frac{1}{2}\frac{x}{3D^2+4D+1}e^{0x} = \frac{x}{2} \quad \text{--- (2)}$$

III

$$\frac{-1}{2}\frac{1}{D^3+2D^2+D}\cos 2x$$

$$\Rightarrow \frac{-1}{2}\frac{1}{-4D-8+D}\cos 2x$$

$$\Rightarrow \frac{-1}{2}\frac{1}{-3D-8}\cos 2x$$

$$\Rightarrow \frac{1}{2}\frac{1}{3D+8}\times\frac{3D-8}{3D-8}\cos 2x$$

$$\Rightarrow \frac{1}{2}\frac{3D-8}{9D^2-64}\cos 2x$$

$$\Rightarrow \frac{1}{2}\frac{-3\times 2\times\sin 2x-8\cos 2x}{9\times -4-64}$$

$$\Rightarrow \frac{1}{2}\frac{(-6\sin 2x-8\cos 2x)}{-100}$$

$$\Rightarrow \frac{6}{200}\sin 2x+\frac{8}{200}\cos 2x$$

$$\Rightarrow \frac{3}{100}\sin 2x+\frac{1}{25}\cos 2x \quad \text{--- (3)}$$

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$y = (1) + (2) + (3)$ Ans.

$$y = \frac{e^{2x}}{18}\left(x^2 - \frac{7}{3}x + \frac{13}{9}\right) + \frac{x}{2} + \frac{3}{100}\sin 2x + \frac{1}{25}\cos 2x$$

Ans

Some Question On C.F + P.I.

(1) $(D^3+1)y = \sin(2x+3)$

(2) $(D^2+4)y = \cos 2x$

(3) $(D^2-4)y = x^2$

(4) $(D^2-5D+6)y = x$

(5) $(D^3+3D^2+2D)y = x^2$

(6) $(D^2-4D+4)y = x^2$

(7) $(D^2+D-2)y = x+\sin x$

* (8) $(D^2+4)y = x^2$

Ans → (P.I.) $y = x^2/4 - 1/8$

(9) $(D^2+D)y = x^2+2x+4$

Ans → $x^3/3 + 4x$

(10) $(D^2+4D+4)y = x^2+e^x+\cos 2x$

Ans → $\frac{1}{4}\left(x^2+2x+\frac{3}{4}\right) + e^x - \frac{1}{8}\sin 2x$

H.W Imp.
$(D^2+a^2)y = \sec ax$

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(11) $(D^2+4D+4)y = x^2+e^x+\cos 2x$

Ans → $\frac{1}{4}\left(x^2+2x+\frac{3}{4}\right)+e^x-\frac{1}{8}\sin 2x$

Que ⟹ $(D^2-1)y = x\sin x$

Solve ⟹ $\frac{1}{D^2-1}\,x\sin x = \frac{1}{D^2-1}(\text{I.P. of } e^{ix}).x$

Put $D \to (D+i)$

$\frac{\text{I.P. of } e^{ix}}{(D+i)^2-1}.x = \frac{\text{I.P. of } e^{ix}}{D^2+i^2+2Di-1}.x$

$\frac{\text{I.P. of } e^{ix}}{D^2+2Di-2}.x = \frac{\text{I.P. of } e^{ix}}{-2\left(1-\frac{D^2}{2}-Di\right)}.x$

$\frac{\text{I.P. of } e^{ix}}{-2}\left\{1-\left(\frac{D^2}{2}+Di\right)\right\}^{-1}x$

$\frac{\text{I.P. of } e^{ix}}{-2}\left[1+\left(\frac{D^2}{2}+Di\right)+\left(\frac{D^2}{2}+Di\right)^2\right]x$

$\frac{\text{I.P. of } e^{ix}}{-2}\left[x+i\right] = \frac{-1}{2}\text{I.P. of}(\cos x+i\sin x)(x+i)$

$\frac{-1}{2}\text{ I.P. of}\left[x\cos x+i\cos x+xi\sin x-\sin x\right]$

$\frac{-1}{2}\left[\cos x+x\sin x\right]$ Ans

Que ⟹ $(D^2 - 2D+1)y = xe^x\sin x$

$\frac{1}{D^2-2D+1}\,xe^x\sin x$

$\frac{e^x}{(D+1)^2-2(D+1)+1}\,x\sin x$

$\frac{e^x}{D^2+\cancel{2D}+\cancel{1}-\cancel{2D}-\cancel{2}+\cancel{1}}\,x\sin x$

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$$\Rightarrow \frac{e^x}{D^2} x \sin x$$

$$\Rightarrow e^x \iint \underset{u}{x} \underset{v}{\sin x}\, dx\, dx$$

$$\Rightarrow e^x \int \left\{ x(-\cos x) - 1.(-\sin x) \right\} dx$$

$$\Rightarrow e^x \int \left\{ -(\underset{u}{x} \underset{v}{\cos x}) + \sin x \right\} dx$$

$$\Rightarrow e^x \left[-(x \sin x - 1.(-\cos x)) - \cos x \right]$$

$$\Rightarrow e^x \left[-x \sin x - \cos x - \cos x \right]$$

$$\Rightarrow -e^x \left[x \sin x + 2\cos x \right]$$ Ans

Que $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$

Solve → $\frac{8}{D^2 - 4D + 4} e^{2x} x^2 \sin 2x$

$$\Rightarrow \frac{8 e^{2x}}{(D+2)^2 - 4(D+2) + 4} x^2 \sin 2x$$

$$\Rightarrow \frac{8 e^{2x}}{D^2 + \cancel{4D} + \cancel{4} - \cancel{4D} - \cancel{8} + \cancel{4}} x^2 \sin 2x$$

$$\Rightarrow \frac{8 e^{2x}}{D^2} x^2 \sin 2x = 8 e^{2x} \iint \underset{u}{x^2} \underset{v}{\sin 2x}\, dx\, dx$$

$$\Rightarrow 8 e^{2x} \int \left[x^2 \left(\frac{-\cos 2x}{2} \right) - 2x \left(\frac{-\sin 2x}{4} \right) + 2 \left(\frac{\cos 2x}{8} \right) \right] dx$$

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$$\Rightarrow 8e^{2x} \int \left[-\frac{x^2 \cos 2x}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} \right] dx$$

$$\Rightarrow 8e^{2x} \left[-\frac{1}{2} \left\{ x^2 \frac{\sin 2x}{2} - \cancel{2}x \left(-\frac{\cos 2x}{\cancel{2}\,4} \right) + 2\left(-\frac{\sin}{\cancel{4}\,8} \right. \right.\right.$$

$$+ \frac{1}{2} \left\{ x \left(-\frac{\cos 2x}{2} \right) - 1 \cdot \left(-\frac{\sin 2x}{4} \right) \right\} + \frac{\sin 2x}{8} \Big]$$

$$\Rightarrow 8e^{2x} \left[-\frac{x^2 \sin 2x}{4} - \frac{x \cos 2x}{4} + \frac{\sin 2x}{8} - \frac{x \cos 2x}{4} + \frac{\sin 2x}{8} + \frac{\sin 2x}{8} \right]$$

$$\Rightarrow 8e^{2x} \left[-\frac{x^2 \sin 2x}{4} - \frac{x \cos 2x}{2} + 3 \frac{\sin 2x}{8} \right]$$

$$\Rightarrow e^{2x} \left[-2x^2 \sin 2x - 4x \cos 2x + 3 \sin 2x \right]$$

$$\Rightarrow e^{2x} \left[-2x^2 \sin 2x - 4x \cos 2x + 3 \sin 2x \right]$$

H.W.

Que $(D^4 - 1) y = x \sin x$

$e^{ix} = \cos x + i \sin x$

Solve $\frac{1}{D^4 - 1} x \sin x$

$$\Rightarrow \frac{1}{D^4 - 1} \text{ I.P. of } e^{ix} . x \qquad D \to D + i$$

$$\Rightarrow \frac{\text{I.P. of } e^{ix}}{(D+i)^4 - 1} . x = \frac{\text{I.P. of } e^{ix}}{\{(D+i)^2\}^2 - 1^2} x$$

$$\Rightarrow \frac{\text{I.P. of } e^{ix}}{D^4 + 4iD^3 - 6D^2 - 4Di} . x$$

$$\Rightarrow \frac{\text{I.P. of } e^{ix}}{D\{D^3 + 4D^2 i - 6D - 4i\}} . x$$

$$\Rightarrow \frac{1}{D}\left[\frac{\text{I.P. of } e^{ix}}{-4i\left(1-\frac{D^3}{4i}-D^2+\frac{3D}{2i}\right)}\cdot x\right]$$

$$\Rightarrow \frac{1}{D}\left[\frac{\text{I.P. of } e^{ix}}{-4i}\left(1-\left(\frac{D^3}{4i}+D^2-\frac{3D}{2i}\right)\right)^{-1}x\right]$$

$$\Rightarrow \frac{1}{D}\left[\frac{\text{I.P. of } e^{ix}}{-4i}\left\{1+\frac{D^3}{4i}+D^2-\frac{3D}{2i}\right\}\right]x$$

$$\Rightarrow \frac{1}{D}\left[\frac{\text{I.P. of } e^{ix}}{-4i}\left\{x-\frac{3}{2i}\right\}\right]$$

$$\Rightarrow \frac{1}{D}\left[\frac{\text{I.P. of } (\cos x+i\sin x)}{-4i}\left(x-\frac{3}{2i}\right)\right]$$

$$\Rightarrow \frac{1}{D}\left[\frac{-1}{4i}\right]\text{I.P. of }(\cos x+i\sin x)(x-3/2i)$$

$$\Rightarrow \frac{-1}{D}\,\text{I.P}\left(\frac{x\cos x}{4i}-\frac{3i\cos x}{2i^2\times 4i}+\frac{xi\sin x}{4i}-\frac{3}{2}\frac{\sin x}{4i}\right)$$

$$\Rightarrow \frac{-1}{D}\left[\frac{3}{2}\cos x+\frac{x\sin x}{4}\right]$$

$$\Rightarrow \frac{-1}{4i}\left[\frac{3}{2}\sin x+x(-\cos x)-(-\sin x)\right]$$

$$\Rightarrow \frac{-1}{4i}\left[\frac{3}{2}\sin x-x\cos x+\sin x\right] \qquad \frac{3}{2}+1 \quad \frac{3+2}{2}$$

$$\Rightarrow \frac{-1}{4i}\left[\frac{5}{2}\sin x-x\cos x\right]$$

$$\Rightarrow -\frac{1}{D}\,\text{I.P}\left[\frac{-x\cos x\, i}{4}+\frac{3}{8}\cos x+\frac{x\sin x}{4}+\frac{3i\sin x}{8}\right]$$

$$\Rightarrow \frac{1}{D}\left[\frac{x\cos x}{4}-\frac{3}{8}\sin x\right]$$

$$\Rightarrow \frac{1}{8}x^2\cos x-\frac{3x\sin x}{8}$$

$$\Rightarrow \frac{x^2\sin x}{4}+\frac{x\cos x}{2}-\frac{\sin x}{8}\ \text{Ans}$$

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Que → $(D^3 + 2D^2 + D)y = x^2 e^{2x} + \sin^2 x$

Que → $(D^2 + 2D + 1)y = x \cos x$

Que → $(D^2 - 1)y = x \sin 3x$

Que → $(D^2 + a^2)y = \sec ax$

↳ By General Method.

Variation Of Parameter

Let, we consider IInd order diffn Eqn

$$y'' + Py' + qy = X$$

Where, P, q and x are constant or fun of x.

Step ① first we will calculate C.F.

let C.F $y = c_1 y_1 + c_2 y_2$

Step ② Calculate Wronskian of y_1, y_2

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Step ③ Now calculate P.I. By using This formula,

$$P.I. \rightarrow -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

This Method is known as "Variation of parameter".

⟹ This Method is applicable only for second order diffn eqn.

This Method is used to find the P.I. of Non Homo. Linear eqn in which the constants in the complementary function are considered to be fun's of the independent var.

① $(D^2+1)y = \operatorname{cosec} x$

C.F. $(D^2+1)y = 0$

step ① $D^2+1 = 0 \Rightarrow D = \pm i$

C.F. $\Rightarrow y = e^{0x}[C_1 \cos \beta x + C_2 \sin \beta x]$

Here $y_1 = \cos \beta x$, $y_2 = \sin \beta x$

step ② Calculate Wronskian;

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x = 1$$

step ③ Formula;

$$P.I \rightarrow y = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$$

$$y = -\cos x \int \frac{\sin x \cdot \operatorname{cosec} x}{1} dx + \sin x \int \frac{\cos x \cdot \operatorname{cosec} x}{1} dx$$

$$= -\cos x \cdot \int 1\, dx + \sin x \int \cot x\, dx$$

$$= -\cos x \cdot x + \sin x \cdot \log(\sin x)$$

Then Sol^n

$$y = C_1 \cos x + C_2 \sin x - x \cos x + \sin x \cdot \log(\sin x)$$

Ans

Imp

② $y'' - 6y' + 9y = e^{3x}/x^2$

C.F → $(D^2 - 6D + 9)y = 0$

→ $D = \dfrac{6 \pm \sqrt{36-36}}{2} = 3, 3$

$y = (c_1 + c_2 x)e^{3x}$

Then $y_1 = e^{3x}$ & $y_2 = xe^{3x}$

→ $W(y_1, y_2) = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3xe^{3x} + e^{3x} \end{vmatrix}$

$= \cancel{3xe^{6x}} + e^{6x} - \cancel{3xe^{6x}} = e^{6x}$

→ Formula,

P.I $y = -y_1 \int \frac{y_2 x}{W} dx + y_2 \int \frac{y_1 x}{W} dx$

$= -e^{3x} \int \frac{x \cancel{e^{3x}} e^{3x}/x^2}{\cancel{e^{6x}}} dx + xe^{3x} \int \frac{e^{3x} \cdot e^{3x}/x^2}{\cancel{e^{6x}}} dx$

$= -e^{3x} \int 1/x \, dx + xe^{3x} \int \frac{1}{x^2} dx$

$= -e^{3x} \log x + xe^{3x} \left(\frac{x^{-1}}{-1}\right)$

$= -e^{3x} \log x - e^{3x}$

$\Rightarrow -e^{3x}(\log x + 1)$ Ans

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(4) $(D^2+1)y = x\sin x$

C.F. $\rightarrow$ $D^2+1=0 \Rightarrow D=\pm i$

$y = (c_1\cos x + c_2\sin x)$

$y_1 = \cos x \quad , \quad y_2 = \sin x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$= \cos^2 x + \sin^2 x$

$= 1$

P.I.

$$y = -\cos x\int \frac{\sin x \cdot x\sin x}{1}\,dx + \sin x\int \frac{\cos x \cdot x\sin x}{1}\,dx$$

$$y = -\cos x\int x\sin^2 x\,dx + \sin x\int x\sin x\cos x\,dx$$

$$y = -\cos x\int x\left(\frac{1-\cos 2x}{2}\right)dx + \sin x\int \frac{x}{2}\sin 2x\,dx$$

$[2\sin A\cos B = \sin(A+B) + \sin(A-B)]$

$$y = -\cos x\int \frac{x}{2}\,dx + \cos x\int \frac{x\cos 2x}{2}\,dx + \frac{\sin x}{2}\int x\sin 2x\,dx$$

$$= -\frac{\cos x}{2}\times\frac{x^2}{2} + \frac{\cos x}{2}\left[\frac{x\sin 2x}{2} + \frac{\cos 2x}{4}\right] + \frac{\sin x}{2}\left[\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right]$$

$$= -\frac{x^2\cos x}{4} + \frac{x\cos x\sin 2x}{4} + \frac{\cos x\cos 2x}{8} - \frac{x\sin x\cos 2x}{4} + \frac{\sin x\sin 2x}{8}$$

Imp $(D^2+2D+1)y = e^{-x}\log x$ Imp

Ans $y=(c_1+c_2x)e^{-x} + \frac{x^2}{2}e^{-x}\log x - \frac{3}{4}x^2e^{-x}$

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$$\Rightarrow \frac{-x^2\cos x}{4} + \frac{x}{4}\left\{\cos x.\sin 2x - \sin x\cos 2x\right\} + \frac{1}{8}\left\{\cos x.\cos 2x + \sin x.\sin x\right\}$$

$$\Rightarrow -\frac{x^2\cos x}{4} + \frac{x}{4}\sin(2x-x) + \frac{1}{8}\cos(2x-x)$$

$$\Rightarrow -\frac{x^2\cos x}{4} + \frac{x}{4}\sin x + \frac{1}{8}\cos x \quad \text{Ans.}$$

Imp ① $y''-6y'+9y = \frac{e^{3x}}{x^2}$ Q.B

Imp ② $y''+4y = 4\tan 2x$ Q.B

③ $y''+y = x\sin x$ Q.B

④ $y''+y = \tan ax$

⑤ $y''+4y = \tan 2x$

Some Important Questions

Imp ⑥ $y''-2y'+y = e^{x}\log x$ Ans $\rightarrow (c_1+c_2x)+\frac{1}{4}x^2\log x - \frac{3}{4}x^2e^x$

⑤ $y''-2y'+2y = e^{x}\tan x$

C.F. $\rightarrow$ $D^2-2D+2=0$

$$D = \frac{2\pm\sqrt{4-8}}{2}$$

$$= \frac{2\pm 2i}{2} = 1\pm i$$

$$y = e^x\left[c_1\cos x + c_2\sin x\right]$$

$$y_1 = e^x\cos x, \quad y_2 = e^x\sin x$$

$$W = \begin{vmatrix} e^x\cos x & e^x\sin x \\ -e^x\sin x + e^x\cos x & e^x\cos x + e^x\sin x \end{vmatrix}$$

$$= e^{2x}\cos^2 x + \cancel{e^{2x}\sin x\cos x} + e^{2x}\sin^2 x - \cancel{e^{2x}\sin x\cos x}$$

$$= e^{2x} \cancel{+2e^{2x}\sin x\cos x}$$

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$$P.I. \rightarrow -e^x\cos x\int \frac{e^x\sin x \cdot e^x\tan x}{e^{2x}} +$$

$$e^x\sin x\int \frac{e^x\cos x \cdot e^x\tan x}{e^{2x}}$$

$$y = -e^x\cos x\int \sin x \cdot \tan x\, dx + e^x\sin x\int \cos x \tan x\, dx$$

$$= -e^x\cos x\int \frac{\sin^2 x}{\cos x}\,dx + e^x\sin x\int \sin x\, dx$$

$(1-\cos^2 x)$

$$= -e^x\cos x\int (\sec x - \cos x)\,dx - e^x\sin x \cdot \cos x$$

$$= -e^x\cos x\left[\log(\sec x + \tan x) - \sin x\right] - e^x\sin x\cos x$$

$$= -e^x\cos x \cdot \log(\sec x + \tan x) + e^x\cancel{\sin x}\cancel{\cos x} - e^x\sin x\cdot\cancel{\cos x}$$

$$= -e^x\cos x \cdot \log(\sec x + \tan x)$$

Ans

$$\boxed{y'' + 4y = 4\tan 2x}$$

Roots $\rightarrow \pm 2i$

$\underrightarrow{\text{C.F.}}\; y = c_1\cos 2x + c_2 \sin 2x$

$$\underrightarrow{W} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$= 2$

$\underrightarrow{\text{P.I.}}\; -\cos 2x \int \dfrac{\sin 2x \cdot \cancel{4}^{2}\tan 2x}{\cancel{2}}\,dx + \sin 2x \int \dfrac{\cos 2x \cdot \cancel{4}^{2}\tan 2x}{\cancel{2}}\,dx$

$\Rightarrow 2\left[-\cos 2x \int \dfrac{\sin^2 2x}{\cos 2x}\,dx + \sin 2x \int \dfrac{\sin 2x}{\cancel{\cos 2x}}\,dx\right]$

$= 2\left[-\cos 2x \int \dfrac{1-\cos^2 2x}{\cos 2x}\,dx + \sin 2x\left(\dfrac{-\cos 2x}{2}\right)\right]$

$\Rightarrow 2\left[-\cos 2x \int (\sec 2x - \cos 2x)\,dx - \dfrac{\sin 2x \cos 2x}{2}\right]$

$\Rightarrow 2\left[-\cos 2x\left\{\dfrac{\log(\sec 2x + \tan 2x)}{2} - \dfrac{\sin 2x}{2}\right\} - \dfrac{\sin 2x \cdot \cos 2x}{2}\right]$

$\Rightarrow -\dfrac{\cos 2x}{1}\log(\sec 2x + \tan 2x) + \dfrac{\sin 2x \cdot \cancel{\cos 2x}}{4} - \dfrac{\sin 2x \cdot \cancel{\cos 2x}}{4}$

$\Rightarrow -\dfrac{\cos 2x}{1}\log(\sec 2x + \tan 2x)$ //

(2) $(D^2+2D+1)y = e^{-x}\log x$

$D = -1, -1$

$y = (c_1 + c_2 x)e^{-x}$

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & -xe^{-x} + e^{-x} \end{vmatrix} = e^{-2x}$$

$\underrightarrow{P.I.}$ $-e^{-x}\int \frac{xe^{-x}\cdot e^{-x}\log x}{e^{-2x}}\,dx + xe^{-x}\int \frac{\cancel{e^{-x}}\cdot \cancel{e^{-x}}\log x}{\cancel{e^{-2x}}}\,dx$

$\Rightarrow -e^{-x}\underbrace{\int x\log x\,dx}_{(I)} + xe^{-x}\underbrace{\int \log x\,dx}_{(II)}$

(I) $\int \underset{2}{x}\underset{1}{\log x}\,dx = \log x\cdot\frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{2}\,dx$

$= \frac{x^2}{2}\log x - \int \frac{x}{2}\,dx$

$= \frac{x^2}{2}\log x - \frac{x^2}{4}$ —— (I)

(II) $\int \log x\,dx = \int \underset{2}{1}\cdot\underset{1}{\log x}\,dx$

$= \log x\cdot x - \int \frac{1}{\cancel{x}}\cdot\cancel{x}\,dx$

$= x\log x - x$ —— (II)

$-e^{-x}\left\{\frac{x^2}{2}\log x - \frac{x^2}{4}\right\} + xe^{-x}\left\{x\log x - x\right\}$

$\Rightarrow -e^{-x}\left[\left(\frac{x^2}{2}\log x - \frac{x^2}{4}\right) - x^2\log x + x^2\right]$

$\Rightarrow -e^{-x}\left[-\frac{x^2}{2}\log x + \frac{3}{4}x^2\right]$

$\Rightarrow \frac{x^2e^{-x}\log x}{2} - \frac{3}{4}x^2e^{-x}$

H.W (3) similar

$$y'' - 2y' + y = e^x \log x$$

$$D^2 - 2D + 1 = 0$$

$$D = 1, 1$$

C.F. $\rightarrow (c_1 + c_2 x) e^x$

P.I. $\rightarrow$ $W = \begin{vmatrix} e^x & x e^x \\ e^x & x e^x + e^x \end{vmatrix}$

$$= \cancel{x e^{2x}} + e^{2x} - \cancel{x e^{2x}} = e^{2x}$$

P.I. $\rightarrow$ $-e^x \int \frac{x e^x \cdot e^x}{e^{2x}} \log x \, dx +$

$$x e^x \int \frac{e^x \cdot e^x \log x}{e^{2x}} dx$$

$$= -e^x \int \underset{II}{x} \underset{I}{\log x} \, dx + x e^x \int \underset{II}{1} \cdot \underset{I}{\log x} \, dx$$

$$= -e^x \left[\log x \left(\frac{x^2}{2} \right) - \int \frac{1}{x} \left(\frac{x^2}{2} \right) dx \right] +$$

$$x e^x \left[\log x (x) - \int \frac{1}{\cancel{x}} x \, dx \right]$$

$$= -e^x \left[\frac{x^2 \log x}{2} - \frac{x^2}{4} \right] + x e^x \left[x \log x - x \right]$$

$$= -\frac{x^2 e^x \log x}{2} - \frac{x^2 e^x}{2} + x^2 e^x \log x - x^2 e^x$$

$$= \frac{1}{2} x^2 e^x \log x - \frac{3}{4} x^2 e^x$$

H.W $(D^2 + 1) y = \operatorname{cosec} x$

Ans:- $y = c_1 \cos x + c_2 \sin x +$
$\sin x \cdot \log(\sin x) - x \cos x$

Cauchy's Homogeneous Linear Differential Equation

An Equation of the form

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + k_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + k_n y = X \quad \text{— (A)}$$

Where, X – fun of x.

This Equation is said to be Chauchy's Homogeneous Linear diffn eqn.

Step ① $\longrightarrow$ $e^z = x$

then $z = \log x$

Step ② $\longrightarrow$ put

$$x\frac{dy}{dx} = Dy$$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

$$x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

There are many types of Homo-eqns

Ist type: $\rightarrow$ we have seen Ist order Homog. eqn $\rightarrow$

$$\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)} \text{ > same power}$$

IInd Type: $\rightarrow$ In eqn (A). diffn eqn is with variable coefficient, and each term has same order i.e.

$-x^n \frac{d^n y}{dx^n}$,

$-x^{(n-1)} \frac{d^{n-1}y}{dx^{n-1}}$

Now, Solve this linear diffn eqn with constant coefficient ~~eqn~~ solved as before.

$\boxed{\text{Derivation}} \rightarrow$

Let, $e^z = x \Rightarrow \log x = z$ --- ①

Now, $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x}$ put $\left\{\frac{d}{dz} = D\right\}$

$$x\frac{dy}{dx} = Dy$$

Again diffn;

$$\frac{d^2y}{dx^2} = \frac{dy}{dz}\left\{\frac{-1}{x^2}\right\} + \frac{1}{x}\frac{d^2y}{dz^2} \cdot \frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2} \frac{d^2y}{dz^2}$$

$$x^2 \frac{d^2y}{dx^2} = (D^2 - D)y$$

$$D(D-1)(D-2)y = \frac{d^3y}{dx^3}x^3$$

Derivation:— let $e^z = x \Rightarrow z = \log x$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{dy}{dz}\left(\frac{1}{x}\right) \quad \Big| \quad x\frac{dy}{dx} = \frac{dy}{dz}, \quad x\frac{dy}{dx} = Dy$$

Put, $\frac{d}{dz} = D$

Que ①

$$x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^2$$

Let $x = e^z$

then $\log x = z$

$y = f(z)$ * By chauchy's H. L. D. Eqⁿ's Formula;

$$D(D-1)y - 4Dy + 6y = e^{2z}$$

$$(D^2 - D - 4D + 6)y = e^{2z}$$

$$(D^2 - 5D + 6)y = e^{2z}$$

A.E. ⟹ $m^2 - 5m + 6 = 0$

$m = 3, 2$

C.F. ⟹

$$y = c_1e^{3z} + c_2e^{2z}$$

P.I. $\frac{1}{D^2 - 5D + 6}e^{2z}$ put $D = a = 2$

$\frac{1}{4 - 10 + 6}e^{2z}$, Here $f(a) = 0$

Then we will Applying Faluire Case;

$$\frac{z}{2D - 5}e^{2z}$$

Now put $D = a = 2$

$$\frac{z}{4-5}e^{2z} = \frac{ze^{2z}}{-1}$$

Then P.I. :— $-\log x \cdot e^{2\log x} = y$

$$y = -\log x \cdot e^{\log x^2}$$

$$\therefore y = -x^2\log x$$

Complete Solⁿ $= c_1e^{3z} + c_2e^{2z} - x^2\log x$

$$c_1x^3 + c_2x^2 - x^2\log x \quad \text{Ans}$$

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(2) $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = x^4$

Solve - This is Cauchy's L.D.E.

Then, Let, $e^z = x$

$\therefore \log x = z$

$$D(D-1)y - 2Dy - 4y = e^{4z}$$

$$(D^2 - D - 2D - 4)y = e^{4z}$$

$$(D^2 - 3D - 4)y = e^{4z}$$

C.F. $\Rightarrow$ $D^2 - 3D - 4 = 0$

$D = 4, -1$

$y = C_1 e^{4z} + C_2 e^{-z}$

P.I. $\Rightarrow$ $\frac{1}{D^2 - 3D - 4} e^{4z}$ $\quad D = a = 4$

$\Rightarrow \frac{1}{16 - 12 - 4} e^{4z}$

Faluire case

$\Rightarrow \frac{z}{2D - 3} e^{4z}$ $\quad D = a = 4$

$\Rightarrow \frac{z e^{4z}}{8 - 3}$

$\Rightarrow \frac{z e^{4z}}{5}$

Then P.I. $y = \frac{1}{5} \log x . e^{4 \log x}$

$= \frac{1}{5} \log x . x^4$

$C.S = C_1 e^{4z} + C_2 e^{-z} + \frac{x^4}{5} \log x$

$= C_1 x^4 + C_2 x^{-1} + \frac{x^4}{5} \log x$

Ans

V. Imp Q. B.

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(3) $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2$

Solve :- Let $e^z = x \Rightarrow z = \log x$

$D(D-1)y - 3Dy + 4y = (1+e^z)^2$

C.F. $\Rightarrow$ $D^2 - D - 3D + 4 = 0$

$D^2 - 4D + 4 = 0$

$D = 2, 2$

$y = (C_1 + C_2 z)e^{2z}$

P.I. $\Rightarrow$ $\frac{1}{D^2-4D+4}(1+e^z)^2$

$\Rightarrow \frac{1}{D^2-4D+4}(1+e^{2z}+2e^z)$

$\Rightarrow \frac{1}{D^2-4D+4}e^{0z} + \frac{1}{D^2-4D+4}e^{2z} + \frac{2}{D^2-4D+4}e^z$

$\Rightarrow \frac{1}{4} + \frac{1}{4-8+4}e^{2z} + \frac{2}{1-4+4}e^z$

$\downarrow$ F.C.

$\Rightarrow \frac{1}{4} + F.C. + \frac{2e^z}{1}$

$\Rightarrow \frac{1}{4} + \frac{z}{2D-4}e^{2z} + 2e^z$

Again F.C.

$\Rightarrow \frac{1}{4} + \frac{z^2 e^{2z}}{2} + 2e^z$

$\Rightarrow \frac{1}{4} + \frac{1}{2}(\log x)^2 e^{2\log x} + 2e^{\log x}$

$\Rightarrow \frac{1}{4} + \frac{(\log x)^2 . x^2}{2} + 2x$

Then C.S. $y = (C_1 + C_2 \log x)e^{2\log x} + \frac{1}{4} + 2x + \frac{(\log x)^2}{2} \times x^2$

$y = (C_1 + C_2 \log x)x^2 + \frac{1}{4} + 2x + \frac{x^2}{2}(\log x)^2$

Ans

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(4) $$x\frac{d^2y}{dx^2} - \frac{2y}{x} = x + \frac{1}{x^2}$$

Solve :- Let $e^z = x$, $\log x = z$

First we will multiplying by x on both sides;

$$x^2\frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}$$

$$D(D-1)y - 2y = e^{2z} + \frac{1}{e^z} \quad \text{--- (1)}$$

C.F. $\rightarrow$ $(D^2 - D - 2) = 0$

$D = -1, 2$

$$y = c_1e^{-z} + c_2e^{2z} = c_1e^{-\log x} + c_2e^{2\log x}$$

$$\Rightarrow \frac{1}{D^2-D-2}\left(e^{2z} + \frac{1}{e^z}\right)$$

$$\Rightarrow \frac{1}{D^2-D-2}e^{2z} + \frac{1}{D^2-D-2}\left(\frac{1}{e^z}\right)$$

$D = -1$

$$\Rightarrow \frac{1}{4-2-2}e^{2z} + \frac{1}{1+1-2}e^{-z}$$

$\hookrightarrow$ F.C. $\quad\hookrightarrow$ F.C

$$\cancel{\Rightarrow \frac{-1}{2}e^{z} + \frac{1}{4}e^{-2z}}$$

$$\boxed{\cancel{y = c_1x^{-1} + c_2x^2 - \frac{1}{2}x + \frac{1}{4}x^{-2}}}$$

$$\Rightarrow \frac{z}{2D-1}e^{2z} + \frac{z}{2D-1}e^{-z}$$

$$\Rightarrow \frac{z}{3}e^{2z} + \frac{z}{-3}e^{-z}$$

Then P.I. :- $$\frac{\log x}{3}x^2 - \frac{\log x}{3}x^{-1}$$

C.S. $\rightarrow$ $$y = c_1x^{-1} + c_2x^2 + \frac{x^2}{3}\log x - \frac{\log x}{3x}$$

Ans

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(5) $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}$

Solve → $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 12\log x$

(multiplying By x^2)

let, $x = e^z \Rightarrow \log x = z$

$$D(D-1)y + Dy = 12z \quad \text{----} (1)$$

C.F.

$$(D^2 - D + D)y = 0$$

$$\Rightarrow D^2 = 0 \Rightarrow D = 0, 0$$

$$y = (C_1 + C_2 z)e^{0z}$$

$$= (C_1 + C_2 z) = (C_1 + C_2 \log x)$$

P. I.

$$\frac{1}{D^2} 12z$$

$$\Rightarrow \iint 12z\, dz = \int 12\frac{z^2}{2} dz$$

$$\Rightarrow \int 6z^2\, dz = 6\frac{z^3}{3} = 2z^3$$

$$\Rightarrow 2(\log x)^3$$

Then

$$C.S. = (C_1 + C_2\log x) + 2(\log x)^3 \quad \text{Ans}$$

V.V. Imp. B.

Que Q. $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10(x + 1/x)$

Ans $5x + 2\left(\frac{\log x}{x}\right)$ A

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(7) $$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2\log x$$

Let, $e^z = x \Rightarrow z = \log x$

Solve $\Rightarrow$ $D(D-1)y - 2Dy - 4y = e^{2z} + 2z$

C.F. $\rightarrow$ $D^2 - D - 2D - 4 = 0$

$D^2 - 3D - 4 = 0$

$D = 4, -1$

$y = c_1 e^{4z} + c_2 e^{-z} = c_1 e^{4\log x} + c_2 e^{-\log x}$

P.I. $\rightarrow$ $\frac{1}{D^2 - 3D - 4}(e^{2z} + 2z)$

$\Rightarrow \frac{1}{D^2-3D-4}e^{2z} + \frac{2}{D^2-3D-4}z$

$\Rightarrow \frac{1}{4-6-4}e^{2z} + \frac{2}{-4\left[1 - \frac{D^2}{4} + \frac{3}{4}D\right]}z$

$\Rightarrow \frac{1}{-6}e^{2z} - \frac{1}{2}\left[1 - \left(\frac{D^2}{4} - \frac{3}{4}D\right)\right]^{-1}z$

$\Rightarrow -\frac{e^{2z}}{6} - \frac{1}{2}\left[1 + \left(\frac{D^2}{4} - \frac{3}{4}D\right) + \ldots\right]z$

$\Rightarrow -\frac{e^{2z}}{6} - \frac{z}{2} + \frac{1}{2} \times \frac{3}{4}$

$\Rightarrow -\frac{1}{6}x^2 - \frac{\log x}{2} + \frac{3}{8}$

C.S. $\rightarrow$ $y = c_1x^4 + c_2x^{-1} - \frac{x^2}{6} - \frac{\log x}{2} + \frac{3}{8}$

[1 MORE que is in the last page of topic ~~Legendre's~~ Symmult. L.d.eq^n] Ans

Q4 → $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x+\frac{1}{x}\right)$

$e^z = x \Rightarrow z = \log x$

$D(D-1)(D-2)y + 2D(D-1)y + 2y = 10\left[e^z + \frac{1}{e^z}\right]$

$(D^2-D)(D-2)$

$D^3 - \cancel{2D^2} - D^2 + \cancel{2D} + \cancel{2D^2} - \cancel{2D} + 2 = -"-$

$D^3 - D^2 + 2 = 10\left[e^z + \frac{1}{e^z}\right]$

P.I. $\frac{10}{D^3-D^2+2}\left\{e^z + e^{-z}\right\}$

$\frac{10}{\cancel{1}-\cancel{1}+2}e^z + \frac{10}{-1-1+2}e^{-z}$

$5e^z + \frac{10z}{3D^2-2D}e^{-z}$

$= 5e^z + \frac{10z}{3+2}e^{-z}$

$= 5e^z + \frac{10z}{5}e^{-z}$

$= 5e^z + 2ze^{-z}$

$= \cancel{5\log}\ 5x + 2\log x \cdot e^{-\log x}$

$= 5x + \frac{2}{x}\log x$

Q. $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 65\cos(\log x)$

* Legendre's Linear D. Equation *

Standard form :-

$$(ax+b)^n \frac{d^n y}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + k_2 (ax+b)^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots$$

$$\cdots + k_n y = X$$

Step ① $\longrightarrow$ put $(ax+b) = e^z$

then $z = \log(ax+b)$

Step ② $\longrightarrow$ put $a^3 D(D-1)(D-2)y = (ax+b)^3 \frac{d^3 y}{dx^3}$

$$a^2 D(D-1)y = (ax+b)^2 \frac{d^2 y}{dx^2}$$

Constant Terms are also vary

$$a\,Dy = (ax+b)\frac{dy}{dx}$$

$$\vdots \qquad\qquad \vdots$$

Then Solve the d.Eⁿ By previous Methods.

$$C.S = C.F + P.I.$$

Que ① $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y$
$= 4\cos\log(1+x)$

Solve $\rightarrow$ let $(1+x) = e^z$
$\Rightarrow x = e^z - 1$
& $\log(1+x) = z$ $\Big\}$ — ①

Now,
$1^2 D(D-1)y + 1 \cdot Dy + y = 4\cos z$

C.F. $\Rightarrow$ $D^2 - \cancel{D} + \cancel{D} + 1 = 0$
$D^2 + 1 = 0$
$D = \pm i$

Then $y = c_1 \cos z + c_2 \sin z$ — ②

P.I. $\Rightarrow$
$\frac{4}{D^2+1}\cos z$

Put $D^2 = -a^2$
$= -1$

But This is F.C.
Then $\frac{4z}{2D}\cos z$

Now we will Solve By ~~Rationali~~ zation Integral-

$\frac{4z}{2D+1} \times \frac{2D-1}{2D-1}\cos z$

$\frac{4z(2D-1)}{4D^2-1}\cos z$

Now put $D^2 = -1$

$\frac{4z(2D-1)}{-4-1}\cos z$

$\frac{4z[-2\sin z - \cos z]}{-5}$

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$$\Rightarrow \cancel{\frac{-4}{5} z \left[-2\sin z - \cos z\right]}$$

$$\Rightarrow \cancel{\frac{-4}{5} \log(1+x) \left[-2\sin(\log(1+x)) - \cos\log(1+x)\right]}$$

$$\frac{4z}{2} \int \cos z \, dz$$

$$2z \sin z$$

again put $z = \log(1+x)$

$$2\log(1+x)\sin(\log(1+x)) \quad \text{---} \ (3)$$

C.S $\Rightarrow$ $y = (2) + (3)$

$$= c_1 \cos \log(1+x) + c_2 \sin \log(1+x) + 2\log(1+x)\sin\log(1+x)$$

Ans.

Imp

Que (2) $\Rightarrow$ $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$

Solve $\Rightarrow$ Let,

$$\left.\begin{aligned} (3x+2) &= e^z \\ \Rightarrow x &= \frac{e^z - 2}{3} \\ \& \quad \log(3x+2) &= z \end{aligned}\right\} \text{---} (1)$$

$$(3)^2 D(D-1)y + 3\times 3 Dy - 36y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1$$

C.F $\Rightarrow$ $9D^2 \cancel{-9D} + \cancel{9D} - 36 = 0$

$$\cancel{D^2 + 2D - 36 = 0}$$

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$$\cancel{D^2 + 2D - 36 = 0}$$

$$\cancel{D = \frac{-2 \pm \sqrt{4+144}}{2}}$$

$$\cancel{= \frac{-2 \pm \sqrt{148}}{2}}$$

$$\cancel{= \frac{-2 \pm 2\sqrt{37}}{2} = -1 \pm \sqrt{37}}$$

$$\cancel{\text{C.F.} \rightarrow y = c_1 e^{(-1+\sqrt{37})z} + c_2 e^{(-1-\sqrt{37})z}} \quad \text{---} (2)$$

$$\cancel{\text{P.I.} \Rightarrow \frac{1}{D^2 + 2D - 36}}$$

$$\frac{e^{2z} + 4 - 4e^{z}}{3} + \frac{4e^{z} - 8}{3} + 1$$

$$\frac{e^{2z} + 4 - \cancel{4e^{z}} + \cancel{4e^{z}} - 8 + 3}{3}$$

$$\frac{e^{2z} - 1}{3}$$

C.F.

A.E. $(9D^2 - 36)y = 0$

$$9D^2 - 36 = 0$$

$$9[D^2 - 4] = 0$$

$$D = \pm 2$$

$$y = c_1 e^{2z} + c_2 e^{-2z}$$

$$\text{P.I.} \rightarrow \frac{1}{9(D^2-4)} \left[3\left(\frac{e^{z}-2}{3}\right)^2 + 4\left(\frac{e^{z}-2}{3}\right) + 1 \right]$$

$$\frac{1}{9(D^2-4)} \left\{ \frac{e^{2z}}{3} - \frac{1}{3} \right]$$

$$\frac{1}{27} \left[\frac{1}{D^2-4} e^{2z} - \frac{1}{D^2-4} \cdot e^{0z} \right]$$

F.C

$$\frac{1}{27} \left(\frac{z}{2D} e^{2z} + \frac{1}{4} \right)$$

$$\frac{1}{27} \left[\frac{z}{2} \frac{e^{2z}}{2} + \frac{1}{4} \right]$$

$$\frac{1}{27} \left(\frac{z e^{2z}}{4} + \frac{1}{4} \right) = \frac{1}{108} \left[\log(3x+2) e^{2\log(3x+2)} + 1 \right.$$

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Then

$$y = c_1 e^{2\log(3x+2)} + c_2 e^{-2\log(3x+2)} + \frac{1}{108}\left[\log(3x+2)\cdot (3x+2)^2 + 1\right]$$

i.e. $$y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}\left[(3x+2)^2\log(3x+2) + 1\right]$$ Ans.

① $$x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 65\cos(\log x)$$

② $$(2x+3)^2\frac{d^2y}{dx^2} - (2x+3)\frac{dy}{dx} - 12y = 6x$$

③ $$(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$$

Imp

Ans ① → $y = c_1x^{-2} + x[c_2\cos(\sqrt{3}\log x) + c_3\sin(\sqrt{3}\log x)] + 8\cos(\log x) - \sin(\log x)$

Ans ② → $y = c_1(2x+3)^a + c_2(2x+3)^b$ where, $a,b = \frac{3\pm\sqrt{57}}{4}$
$- \frac{3}{14}(2x+3) + \frac{3}{4}$

Ans ③ → $y = c_1\cos[\log(1+x)] + c_2\sin[\log(1+x)] - \log(1+x)\cdot\cos[\log(1+x)]$